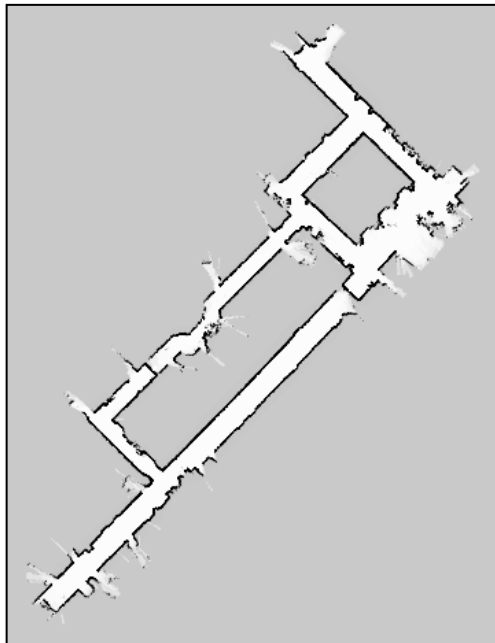


Robot Mapping

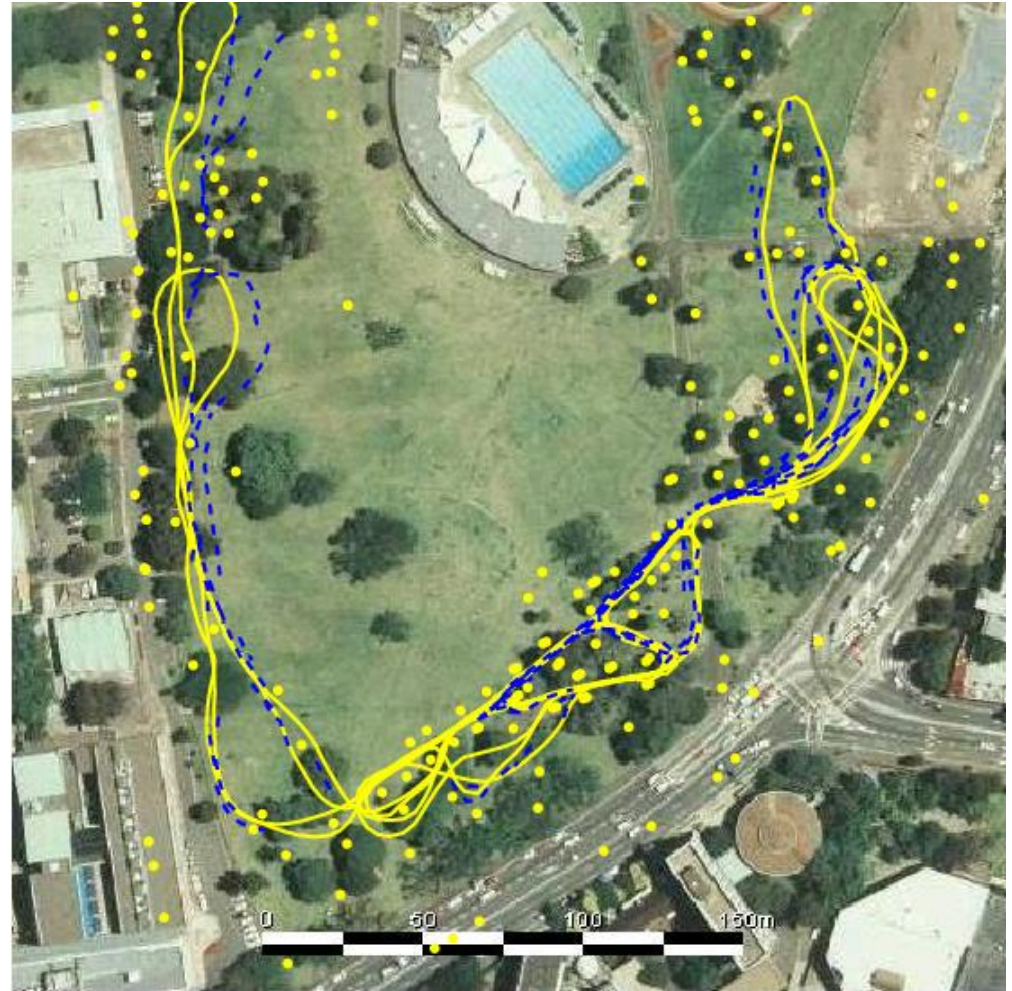
Grid Maps

Gian Diego Tipaldi, Wolfram Burgard

Features vs. Volumetric Maps



Courtesy: D. Hähnel



Courtesy: E. Nebot

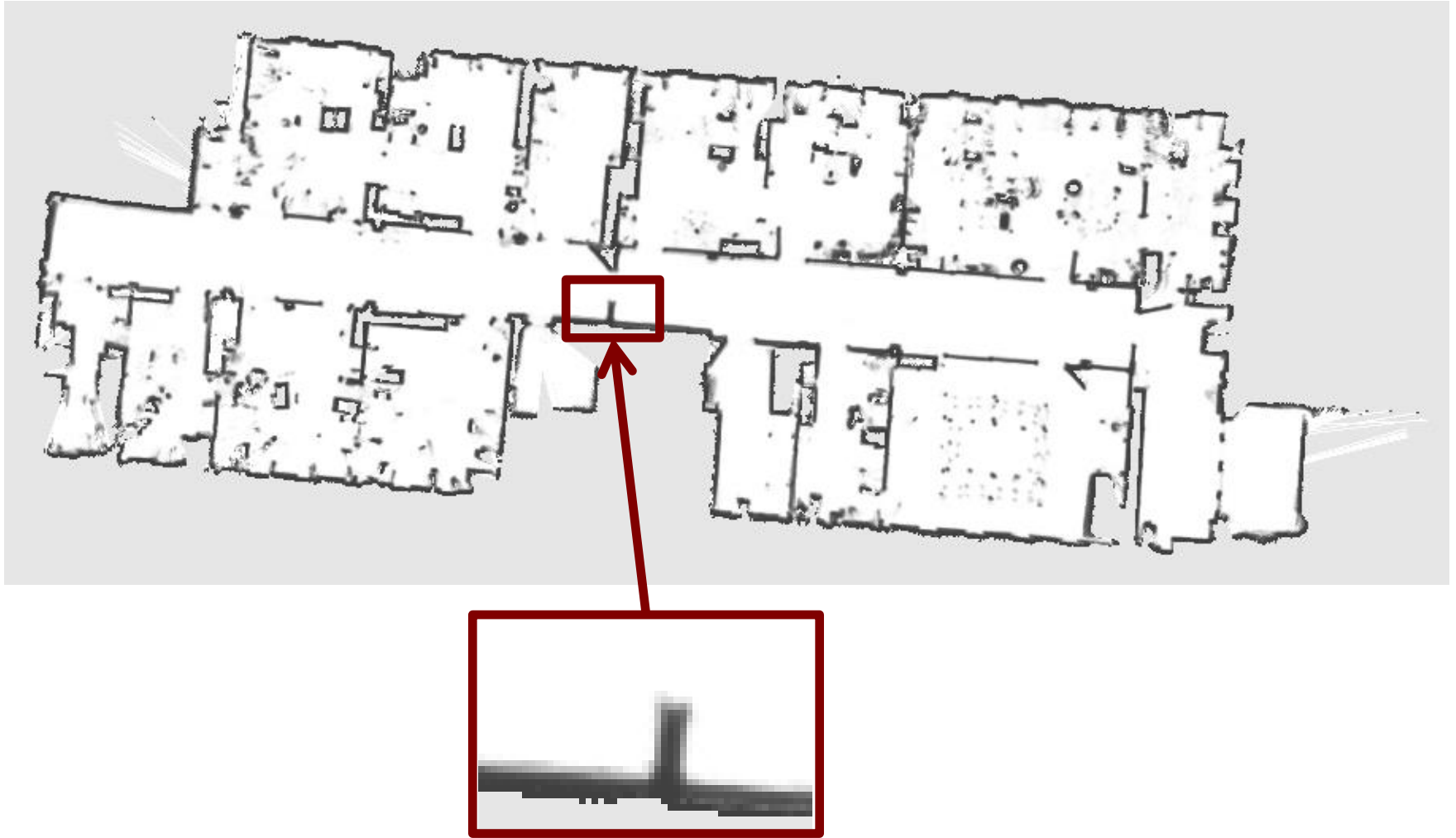
Features

- So far, we only used feature maps
- Natural choice for Kalman filter-based SLAM systems
- Compact representation
- Multiple feature observations improve the landmark position estimate (EKF)

Grid Maps

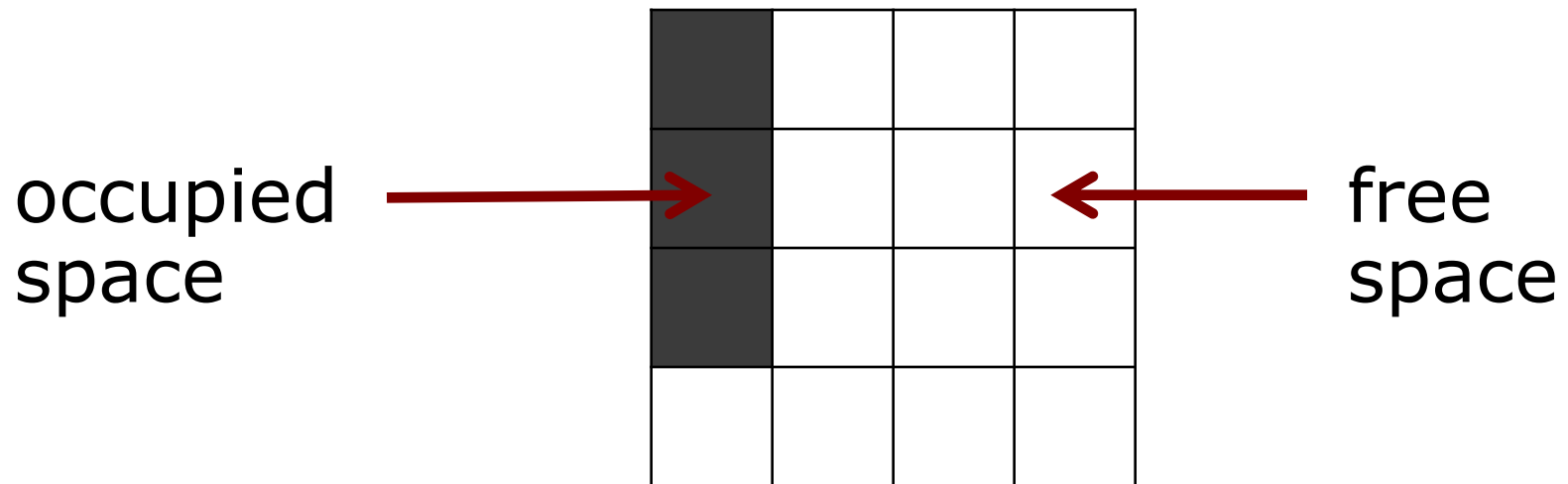
- Discretize the world into cells
- Grid structure is rigid
- Each cell is assumed to be occupied or free space
- Non-parametric model
- Large maps require substantial memory resources
- Do not rely on a feature detector

Example



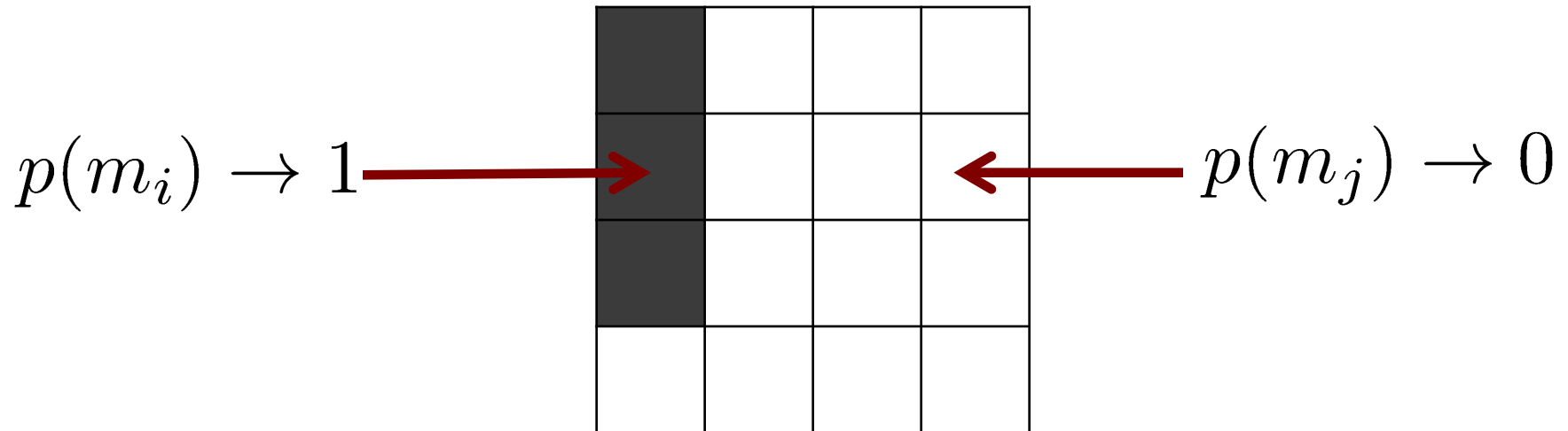
Assumption 1

- The area that corresponds to a cell is either completely free or occupied



Representation

- Each cell is a **binary random variable** that models the occupancy

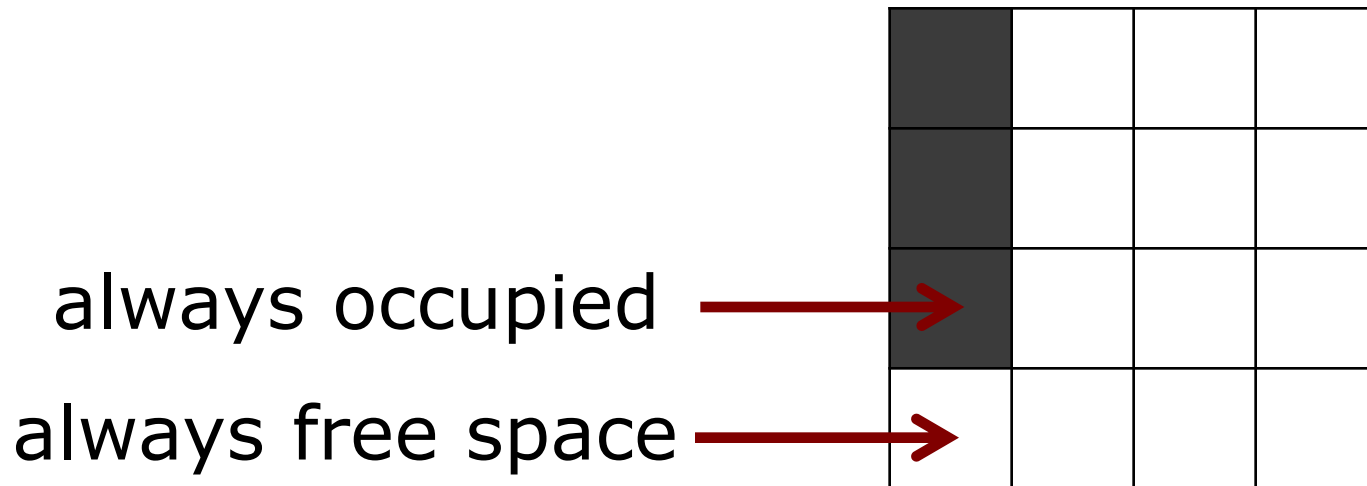


Occupancy Probability

- Each cell is a **binary random variable** that models the occupancy
- Cell is occupied: $p(m_i) = 1$
- Cell is not occupied: $p(m_i) = 0$
- No knowledge: $p(m_i) = 0.5$

Assumption 2

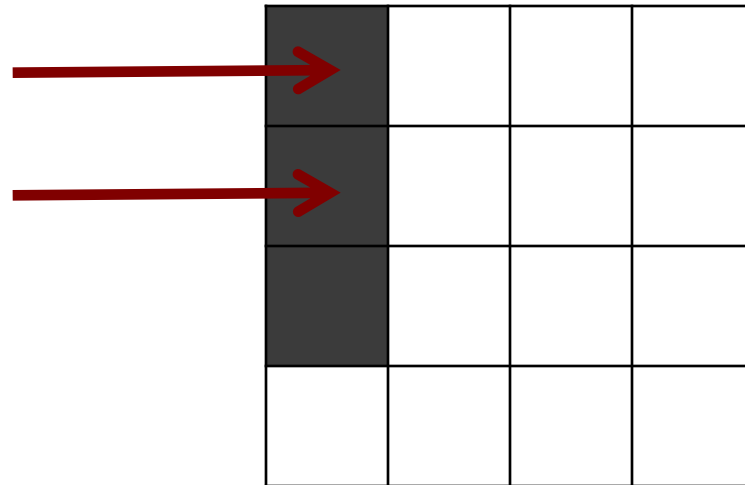
- The world is **static** (most mapping systems make this assumption)



Assumption 3

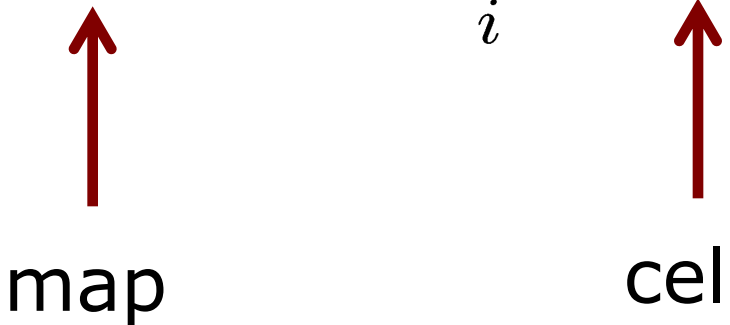
- The cells (the random variables) are **independent** of each other

no dependency
between the cells



Representation

- The probability distribution of the map is given by the product over the cells

$$p(m) = \prod_i p(m_i)$$


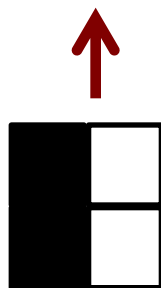
map

cell

Representation

- The probability distribution of the map is given by the product over the cells

$$p(m) = \prod_i p(m_i)$$



example map
(4-dim state)



4 individual cells

Estimating a Map From Data

- Given sensor data $z_{1:t}$ and the poses $x_{1:t}$ of the sensor, estimate the map

$$p(m \mid z_{1:t}, x_{1:t}) = \prod_i p(m_i \mid z_{1:t}, x_{1:t})$$

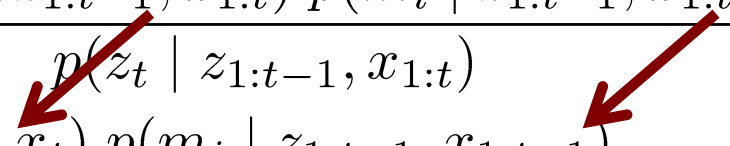
binary random variable

➔ Binary Bayes filter
(for a static state)

Static State Binary Bayes Filter

$$p(m_i \mid z_{1:t}, x_{1:t}) \stackrel{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t}) p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$


Static State Binary Bayes Filter

$$p(m_i | z_{1:t}, x_{1:t}) \stackrel{\text{Bayes rule}}{=} \frac{p(z_t | m_i, z_{1:t-1}, x_{1:t}) p(m_i | z_{1:t-1}, x_{1:t})}{p(z_t | z_{1:t-1}, x_{1:t})}$$
$$\stackrel{\text{Markov}}{=} \frac{p(z_t | m_i, x_t) p(m_i | z_{1:t-1}, x_{1:t-1})}{p(z_t | z_{1:t-1}, x_{1:t})}$$


Static State Binary Bayes Filter

$$p(m_i | z_{1:t}, x_{1:t}) \stackrel{\text{Bayes rule}}{=} \frac{p(z_t | m_i, z_{1:t-1}, x_{1:t}) p(m_i | z_{1:t-1}, x_{1:t})}{p(z_t | z_{1:t-1}, x_{1:t})}$$


$$\stackrel{\text{Markov}}{=} \frac{p(z_t | m_i, x_t) p(m_i | z_{1:t-1}, x_{1:t-1})}{p(z_t | z_{1:t-1}, x_{1:t})}$$

$$p(z_t | m_i, x_t) \stackrel{\text{Bayes rule}}{=} \frac{p(m_i | z_t, x_t) p(z_t | x_t)}{p(m_i | x_t)}$$


Static State Binary Bayes Filter

$$p(m_i | z_{1:t}, x_{1:t}) \stackrel{\text{Bayes rule}}{=} \frac{p(z_t | m_i, z_{1:t-1}, x_{1:t}) p(m_i | z_{1:t-1}, x_{1:t})}{p(z_t | z_{1:t-1}, x_{1:t})}$$
$$\stackrel{\text{Markov}}{=} \frac{p(z_t | m_i, x_t) p(m_i | z_{1:t-1}, x_{1:t-1})}{p(z_t | z_{1:t-1}, x_{1:t})}$$
$$\stackrel{\text{Bayes rule}}{=} \frac{p(m_i | z_t, x_t) p(z_t | x_t) p(m_i | z_{1:t-1}, x_{1:t-1})}{p(m_i | x_t) p(z_t | z_{1:t-1}, x_{1:t})}$$

Static State Binary Bayes Filter

$$\begin{array}{l}
 p(m_i \mid z_{1:t}, x_{1:t}) \\
 \text{Bayes rule} \\
 \text{Markov} \\
 \text{Bayes rule} \\
 \text{indep.}
 \end{array}
 \quad
 \begin{array}{l}
 \underline{\underline{=}} \\
 \underline{\underline{=}} \\
 \underline{\underline{=}} \\
 \underline{\underline{=}}
 \end{array}
 \quad
 \begin{array}{l}
 \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t}) p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \frac{p(z_t \mid m_i, x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \frac{p(m_i \mid z_t, x_t) p(z_t \mid x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i \mid x_t) p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \frac{p(m_i \mid z_t, x_t) p(z_t \mid x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i) p(z_t \mid z_{1:t-1}, x_{1:t})}
 \end{array}$$


Static State Binary Bayes Filter

$$\begin{array}{l}
 p(m_i \mid z_{1:t}, x_{1:t}) \quad \underline{\text{Bayes rule}} \quad \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t}) p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \\
 \quad \underline{\text{Markov}} \quad \frac{p(z_t \mid m_i, x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \\
 \quad \underline{\text{Bayes rule}} \quad \frac{p(m_i \mid z_t, x_t) p(z_t \mid x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i \mid x_t) p(z_t \mid z_{1:t-1}, x_{1:t})} \\
 \\
 \quad \underline{\text{indep.}} \quad \frac{p(m_i \mid z_t, x_t) p(z_t \mid x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i) p(z_t \mid z_{1:t-1}, x_{1:t})}
 \end{array}$$

Do exactly the same for the opposite event:

$$p(\neg m_i \mid z_{1:t}, x_{1:t}) \quad \underline{\text{the same}} \quad \frac{p(\neg m_i \mid z_t, x_t) p(z_t \mid x_t) p(\neg m_i \mid z_{1:t-1}, x_{1:t-1})}{p(\neg m_i) p(z_t \mid z_{1:t-1}, x_{1:t})}$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{p(\neg m_i \mid z_{1:t}, x_{1:t})} = \frac{\frac{p(m_i \mid z_t, x_t) \cancel{p(z_t \mid x_t)} p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i) \cancel{p(z_t \mid z_{1:t-1}, x_{1:t})}}}{\frac{p(\neg m_i \mid z_t, x_t) \cancel{p(z_t \mid x_t)} p(\neg m_i \mid z_{1:t-1}, x_{1:t-1})}{p(\neg m_i) \cancel{p(z_t \mid z_{1:t-1}, x_{1:t})}}}$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\begin{aligned} & \frac{p(m_i \mid z_{1:t}, x_{1:t})}{p(\neg m_i \mid z_{1:t}, x_{1:t})} \\ &= \frac{p(m_i \mid z_t, x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1}) p(\neg m_i)}{p(\neg m_i \mid z_t, x_t) p(\neg m_i \mid z_{1:t-1}, x_{1:t-1}) p(m_i)} \\ &= \frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)} \frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})} \frac{1 - p(m_i)}{p(m_i)} \end{aligned}$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\begin{aligned} & \frac{p(m_i \mid z_{1:t}, x_{1:t})}{1 - p(m_i \mid z_{1:t}, x_{1:t})} \\ &= \frac{p(m_i \mid z_t, x_t) p(m_i \mid z_{1:t-1}, x_{1:t-1}) p(\neg m_i)}{p(\neg m_i \mid z_t, x_t) p(\neg m_i \mid z_{1:t-1}, x_{1:t-1}) p(m_i)} \\ &= \underbrace{\frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)}}_{\text{uses } z_t} \underbrace{\frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})}}_{\text{recursive term}} \underbrace{\frac{1 - p(m_i)}{p(m_i)}}_{\text{prior}} \end{aligned}$$

From Ratio to Probability

- We can easily turn the ration into the probability

$$\frac{p(x)}{1 - p(x)} = Y$$

$$p(x) = Y - Y p(x)$$

$$p(x) (1 + Y) = Y$$

$$p(x) = \frac{Y}{1 + Y}$$

$$p(x) = \frac{1}{1 + \frac{1}{Y}}$$

From Ratio to Probability

- Using $p(x) = [1 + Y^{-1}]^{-1}$ directly leads to

$$p(m_i | z_{1:t}, x_{1:t}) \\ = \left[1 + \frac{1 - p(m_i | z_t, x_t)}{p(m_i | z_t, x_t)} \frac{1 - p(m_i | z_{1:t-1}, x_{1:t-1})}{p(m_i | z_{1:t-1}, x_{1:t-1})} \frac{p(m_i)}{1 - p(m_i)} \right]^{-1}$$

For reasons of efficiency, one performs the calculations in the log odds notation

Log Odds Notation

- The log odds notation computes the logarithm of the ratio of probabilities

$$\frac{p(m_i | z_{1:t}, x_{1:t})}{1 - p(m_i | z_{1:t}, x_{1:t})} = \underbrace{\frac{p(m_i | z_t, x_t)}{1 - p(m_i | z_t, x_t)}}_{\text{uses } z_t} \underbrace{\frac{p(m_i | z_{1:t-1}, x_{1:t-1})}{1 - p(m_i | z_{1:t-1}, x_{1:t-1})}}_{\text{recursive term}} \underbrace{\frac{1 - p(m_i)}{p(m_i)}}_{\text{prior}}$$

$$\rightarrow l(m_i | z_{1:t}, x_{1:t}) = \log \left(\frac{p(m_i | z_{1:t}, x_{1:t})}{1 - p(m_i | z_{1:t}, x_{1:t})} \right)$$

Log Odds Notation

- Log odds ratio is defined as

$$l(x) = \log \frac{p(x)}{1 - p(x)}$$

- and with the ability to retrieve $p(x)$

$$p(x) = \frac{\exp l(x)}{1 + \exp l(x)}$$

Occupancy Mapping in Log Odds Form

- The product turns into a sum

$$\begin{aligned} l(m_i \mid z_{1:t}, x_{1:t}) \\ = \underbrace{l(m_i \mid z_t, x_t)}_{\text{inverse sensor model}} + \underbrace{l(m_i \mid z_{1:t-1}, x_{1:t-1})}_{\text{recursive term}} - \underbrace{l(m_i)}_{\text{prior}} \end{aligned}$$

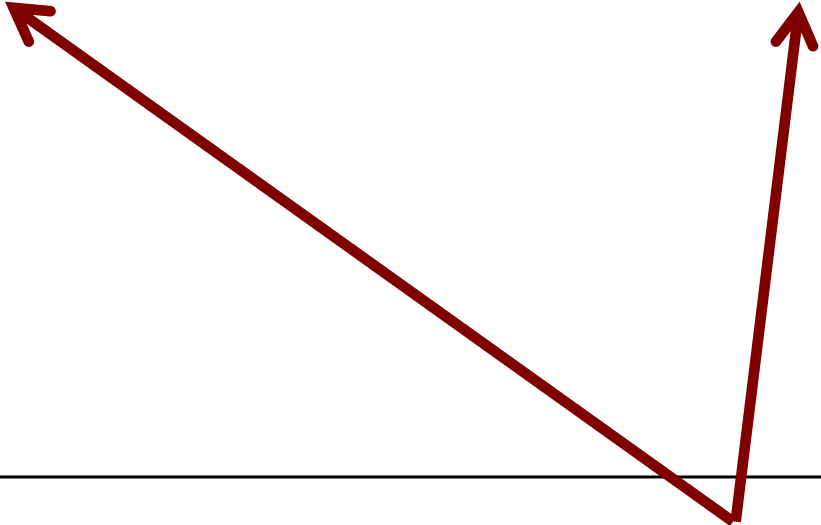
- or in short

$$l_{t,i} = \text{inv_sensor_model}(m_i, x_t, z_t) + l_{t-1,i} - l_0$$

Occupancy Mapping Algorithm

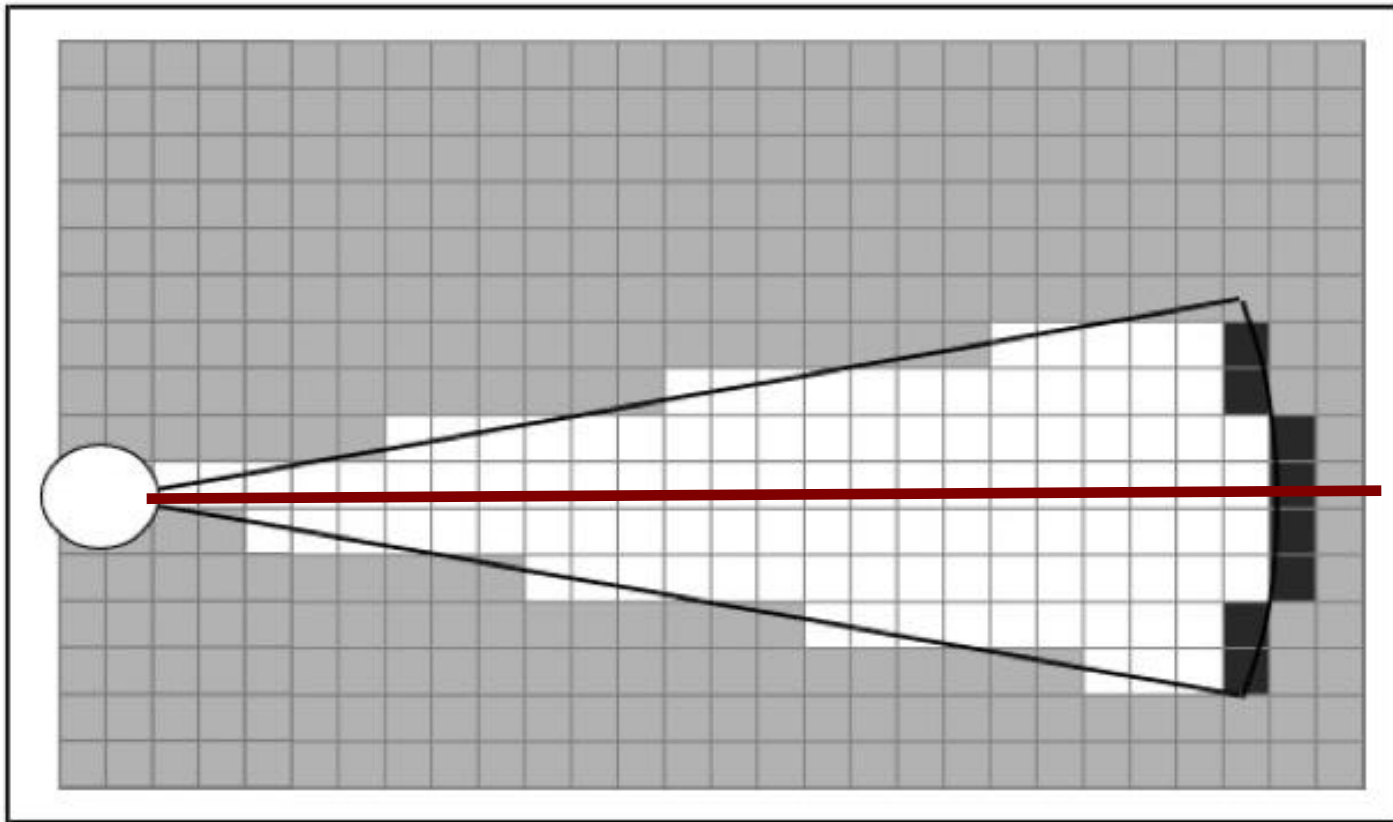
occupancy_grid_mapping($\{l_{t-1,i}\}, x_t, z_t$):

```
1:   for all cells  $m_i$  do
2:     if  $m_i$  in perceptual field of  $z_t$  then
3:        $l_{t,i} = l_{t-1,i} + \text{inv\_sensor\_model}(m_i, x_t, z_t) - l_0$ 
4:     else
5:        $l_{t,i} = l_{t-1,i}$ 
6:     endif
7:   endfor
8:   return  $\{l_{t,i}\}$ 
```



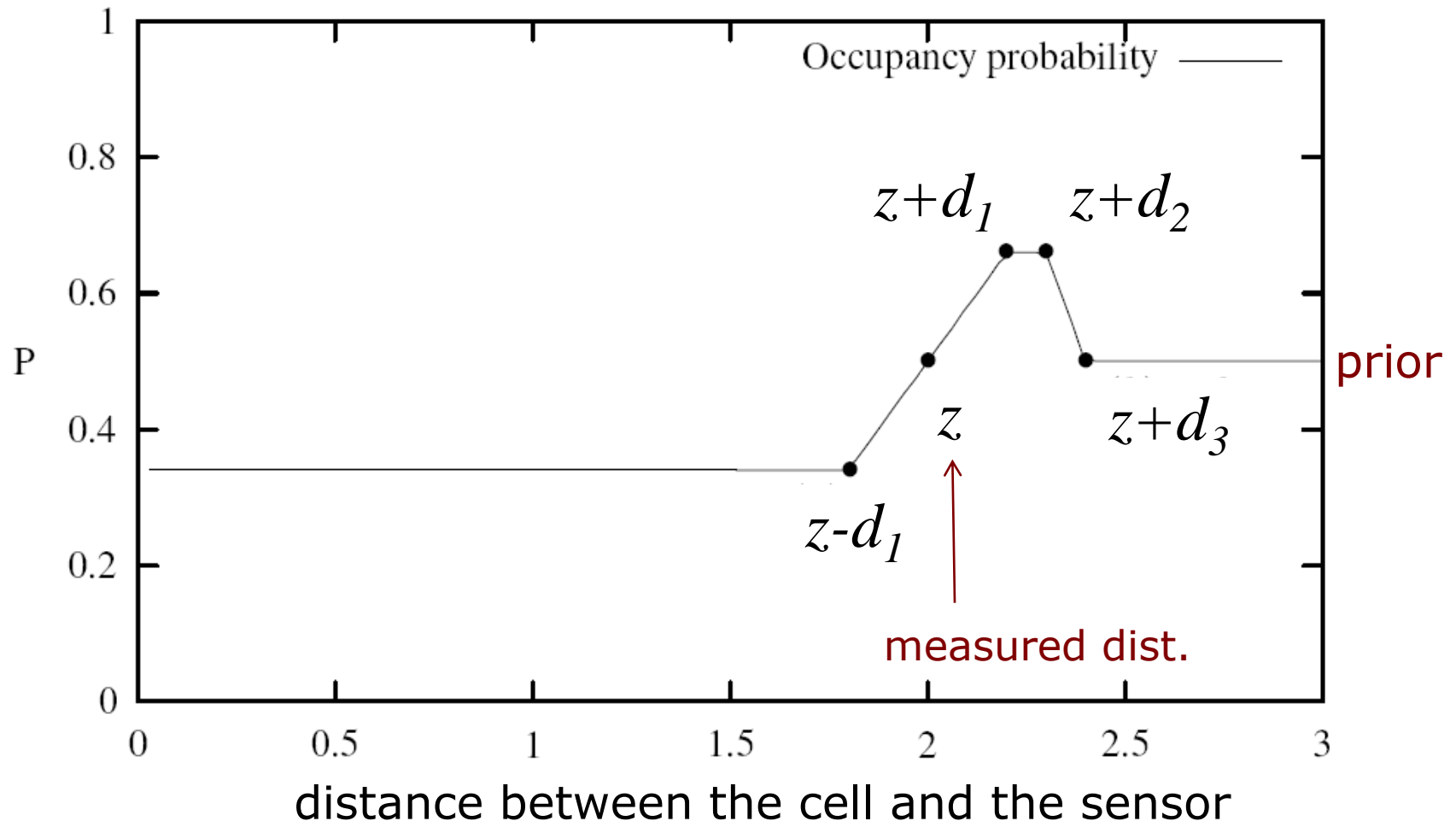
highly efficient, we only have to compute sums

Inverse Sensor Model for Sonar Range Sensors

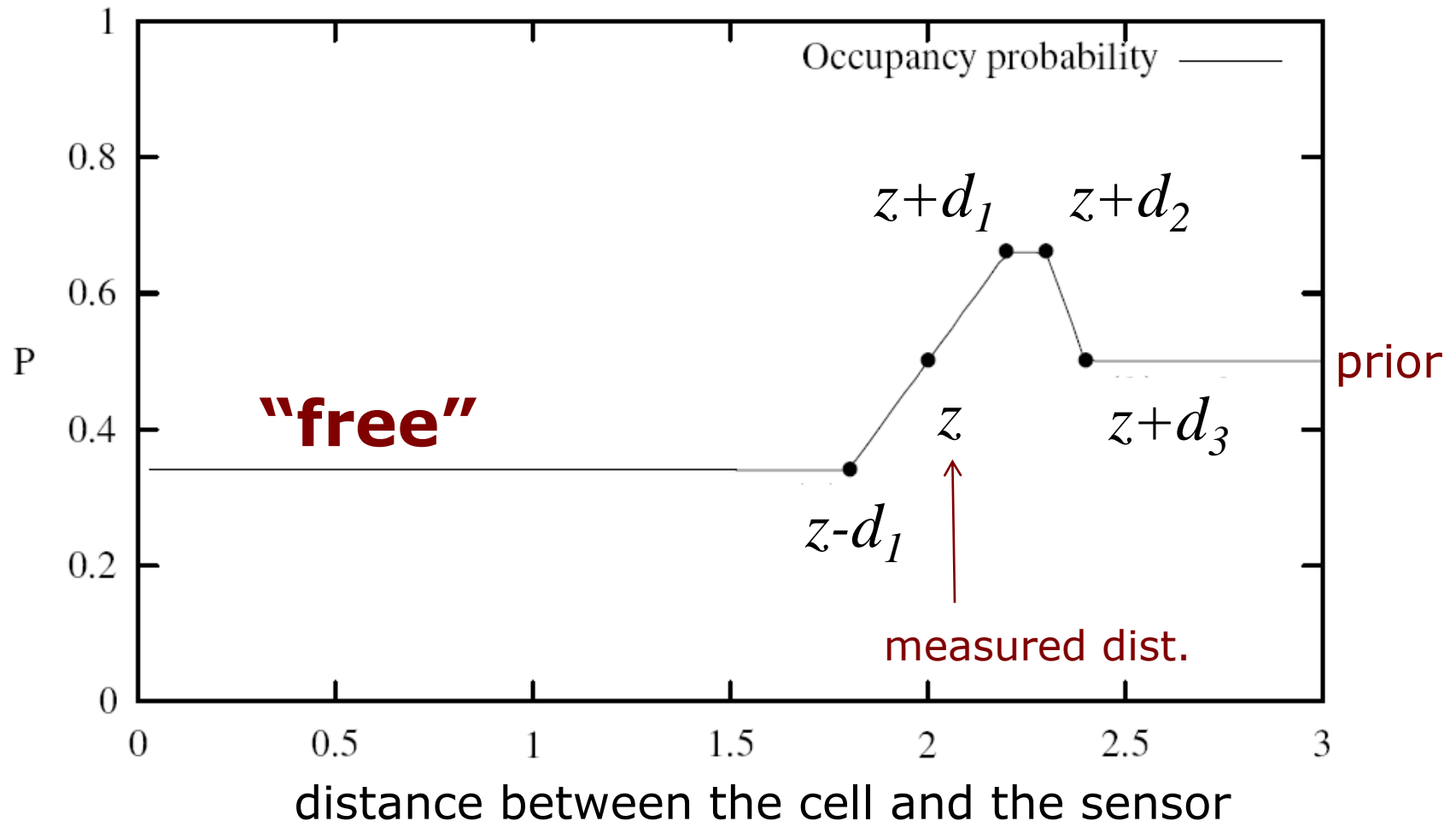


In the following, consider the cells along the optical axis (red line)

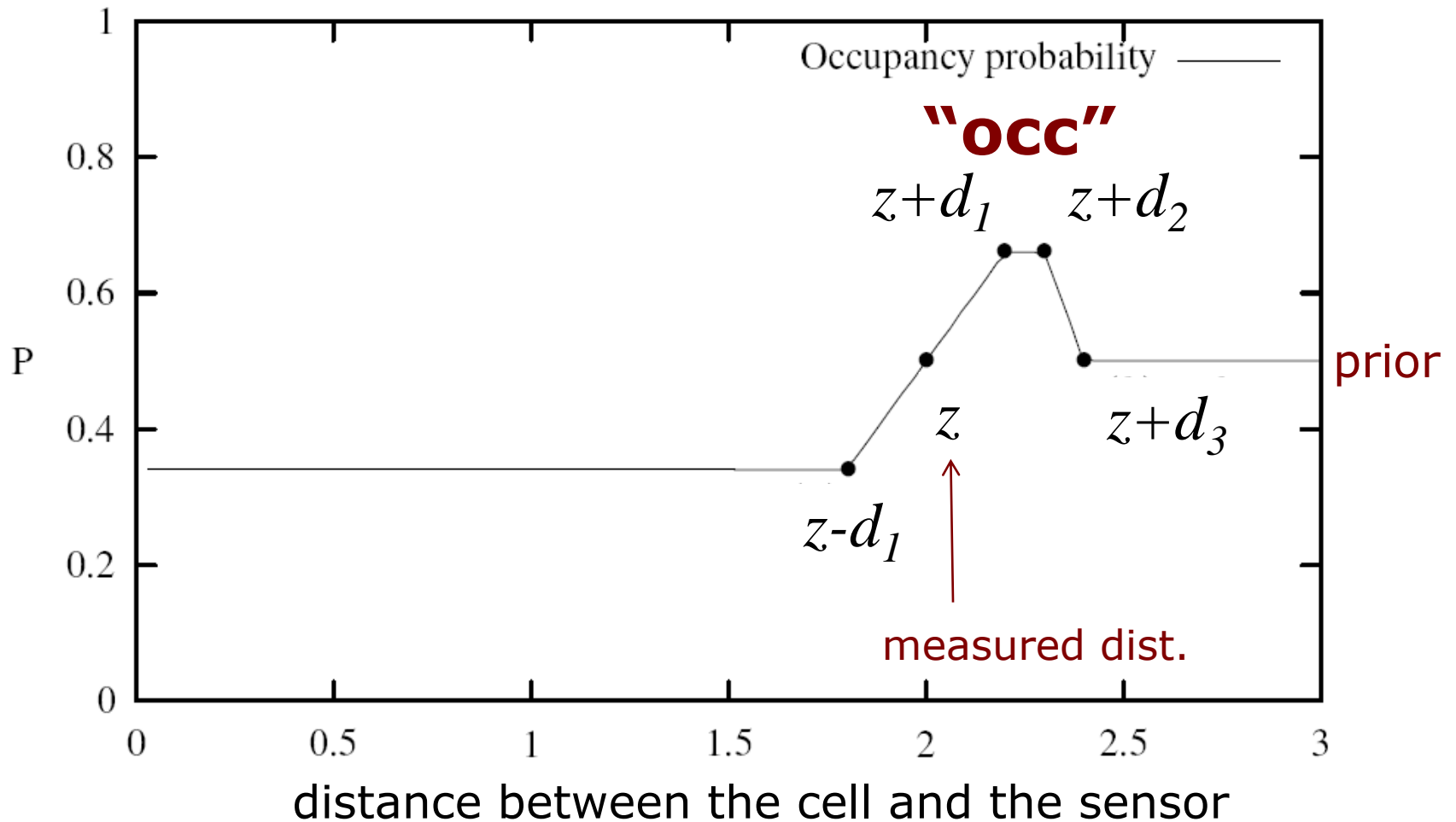
Occupancy Value Depending on the Measured Distance



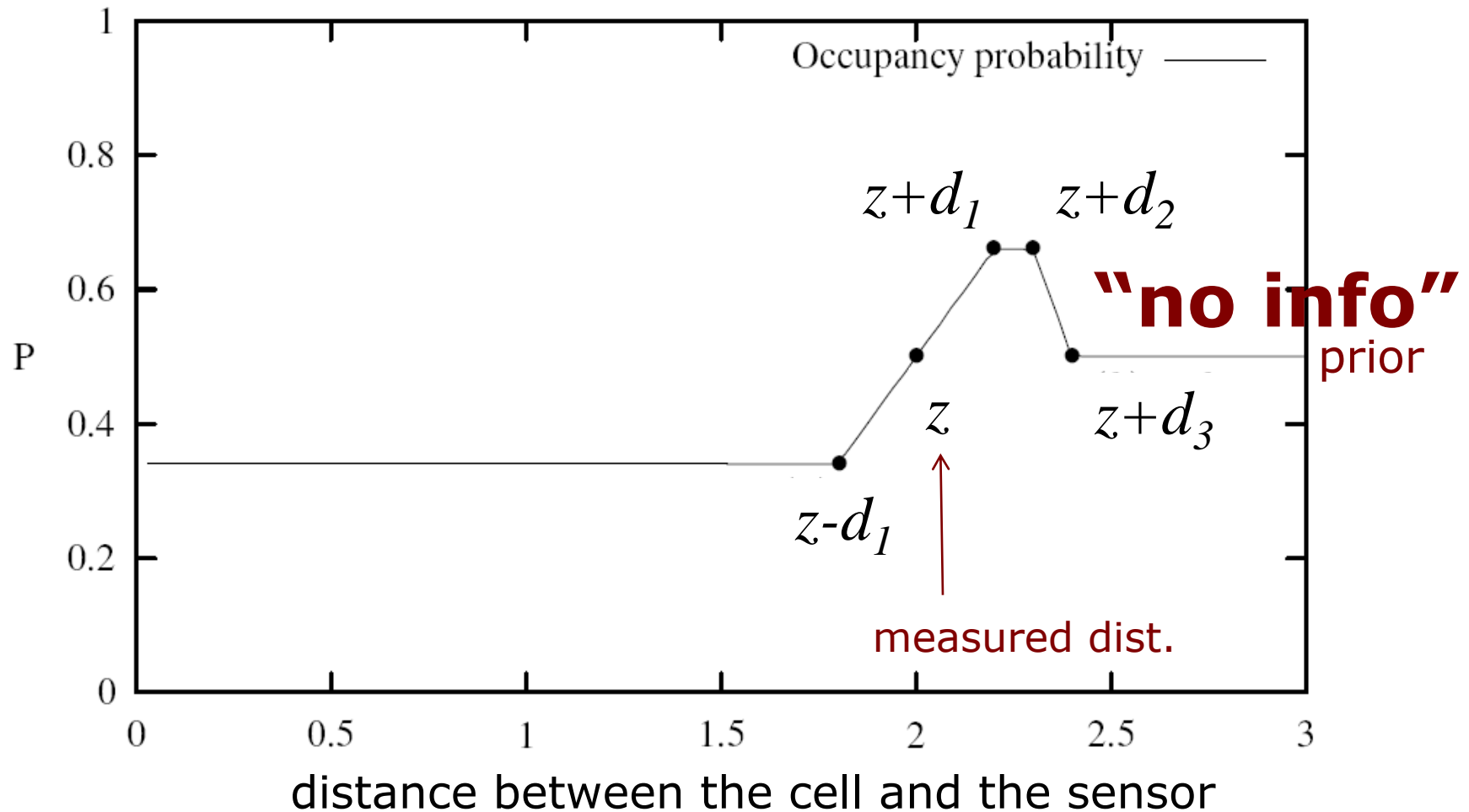
Occupancy Value Depending on the Measured Distance



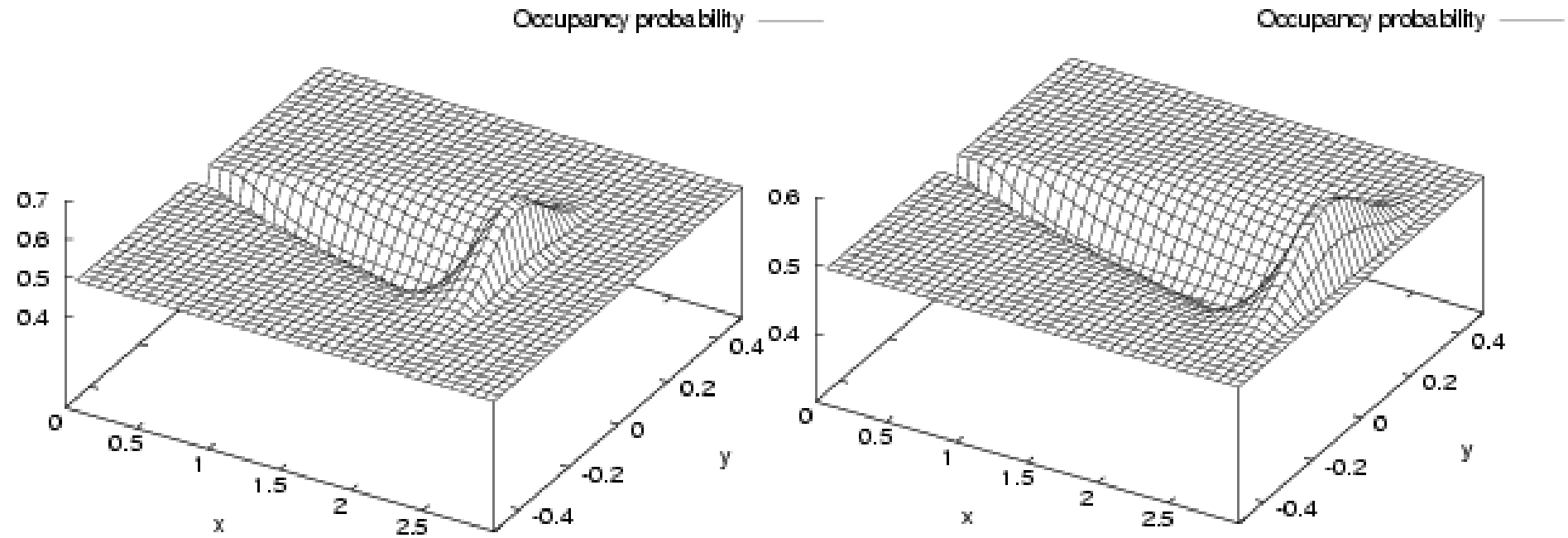
Occupancy Value Depending on the Measured Distance



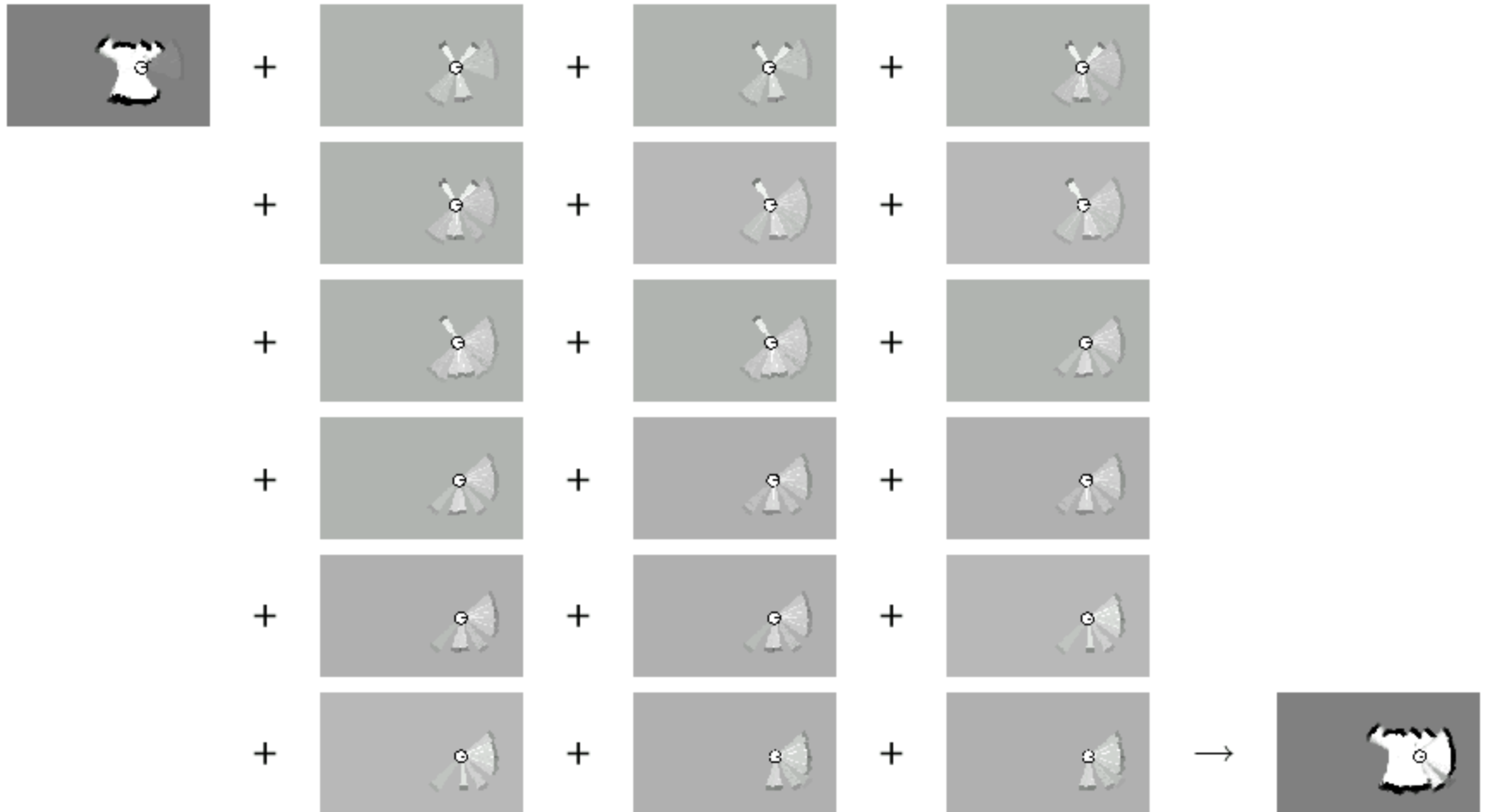
Occupancy Value Depending on the Measured Distance



The Model in More Details



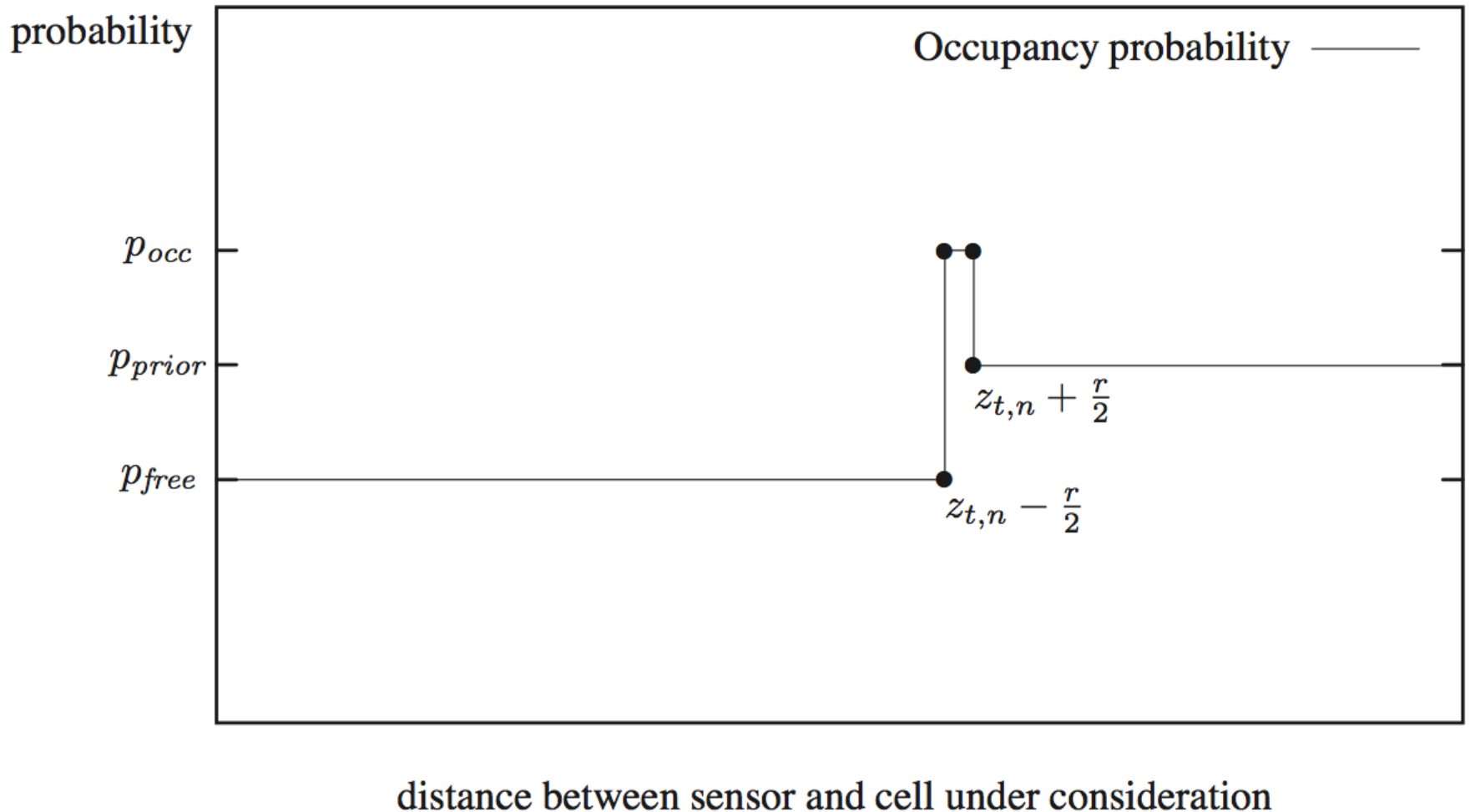
Example: Incremental Updating of Occupancy Grids



Resulting Map Obtained with 24 Sonar Range Sensors



Inverse Sensor Model for Laser Range Finders



Occupancy Grid Mapping

- Moravec and Elfes proposed occupancy grid mapping in the mid 80'ies
- Developed for noisy sonar sensors
- Also called “mapping with know poses”

Occupancy Grid Mapping

- Moravec and Elfes proposed occupancy grid mapping in the mid 80'ies
- Developed for noisy sonar sensors
- Also called “mapping with know poses”
- Lasers are coherent and precise
- Approximate the beam as a “line”

Maximum Likelihood Grid Maps

- Compute values for m that maximize

$$\begin{aligned} m^* &= \operatorname{argmax}_m P(z_1, \dots, z_t \mid m, x_1, \dots, x_t) \\ &= \operatorname{argmax}_m \prod_{t=1}^T P(z_t \mid m, x_t) \quad \text{since } z_t \text{ independent} \\ &\quad \text{and only depend on } x_t \\ &= \operatorname{argmax}_m \sum_{t=1}^T \ln P(z_t \mid m, x_t) \end{aligned}$$

- The individual likelihood are Bernoulli

$$p(z_{t,n} \mid x_t, m) = \begin{cases} \prod_{k=0}^{z_{t,n}-1} (1 - m_{f(x_{t,n},k)}) & \text{if } \zeta_{t,n} = 1 \\ m_{f(x_{t,n},z_{t,n})} \prod_{k=0}^{z_{t,n}-1} (1 - m_{f(x_{t,n},k)}) & \zeta_{t,n} = 0 \end{cases}$$

Maximum Likelihood Grid Maps

- Collecting the terms for each cell:

$$m^* = \operatorname{argmax}_m \sum_{j=1}^J \left(\alpha_j \ln m_j + \beta_j \ln(1 - m_j) \right)$$

- where we have

$$\alpha_j = \sum_{t=1}^T \sum_{n=1}^N I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n})$$

$$\beta_j = \sum_{t=1}^T \sum_{n=1}^N \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j)$$

Maximum Likelihood Grid Maps

- Collecting the terms for each cell:

$$m^* = \operatorname{argmax}_m \sum_{j=1}^J \left(\alpha_j \ln m_j + \beta_j \ln(1 - m_j) \right)$$

- where we have

$$\alpha_j = \sum_{t=1}^T \sum_{n=1}^N I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n}) \quad \boxed{\text{hits}(j)}$$

$$\beta_j = \sum_{t=1}^T \sum_{n=1}^N \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j) \quad \boxed{\text{misses}(j)}$$

- Setting the gradient to zero we obtain

$$\frac{\partial}{\partial m_j} = \frac{\alpha_j}{m_j} - \frac{\beta_j}{1-m_j} = 0 \quad m_j = \frac{\alpha_j}{\alpha_j + \beta_j}$$

Posterior Distribution of Cells

- Maximum likelihood neglects the prior
- We would like to compute

$$\begin{aligned} P(m | x_1, \dots, x_t, z_1, \dots, z_t) &= \\ &= \eta P(z_1, \dots, z_t | m, x_1, \dots, x_t) P(m) \end{aligned}$$

- Likelihood is still Bernoulli
- Conjugate prior: Beta distribution

$$p(m_j; \bar{\alpha}, \bar{\beta}) = \frac{1}{B(\bar{\alpha}, \bar{\beta})} m_j^{\bar{\alpha}-1} (1 - m_j)^{\bar{\beta}-1}$$

Posterior Distribution of Cells

- How does the posterior look like?
- Conjugate prior: Beta distribution

$$p(m_j; \bar{\alpha}, \bar{\beta}) = \frac{1}{B(\bar{\alpha}, \bar{\beta})} m_j^{\bar{\alpha}-1} (1 - m_j)^{\bar{\beta}-1}$$

- Likelihood Bernoulli

$$P(z_1, \dots, z_t \mid m_j, x_1, \dots, x_t) = m_j^{\alpha_j} (1 - m_j)^{\beta_j}$$

- Posterior

$$p(m_j; \hat{\alpha}, \hat{\beta}) =$$



Posterior Distribution of Cells

- How does the posterior look like?
- Conjugate prior: Beta distribution

$$p(m_j; \bar{\alpha}, \bar{\beta}) = \frac{1}{B(\bar{\alpha}, \bar{\beta})} m_j^{\bar{\alpha}-1} (1 - m_j)^{\bar{\beta}-1}$$

- Likelihood Bernoulli

$$P(z_1, \dots, z_t \mid m_j, x_1, \dots, x_t) = m_j^{\alpha_j} (1 - m_j)^{\beta_j}$$

- Posterior

$$p(m_j; \hat{\alpha}, \hat{\beta}) = \frac{1}{B(\bar{\alpha} + \alpha_j, \bar{\beta} + \beta_j)} m_j^{\bar{\alpha} + \alpha_j - 1} (1 - m_j)^{\bar{\beta} + \beta_j - 1}$$

Posterior Estimates of Cells

- Maximum a posteriori (mode of Beta)

$$m^* = \operatorname{argmax}_m P(m \mid x_1, \dots, x_t, z_1, \dots, z_t) = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2}$$

- Expected value (unbiased)

$$m^* = \mathbb{E}[P(m \mid x_1, \dots, x_t, z_1, \dots, z_t)] = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}$$

- Maximum likelihood (revised)

- Maximum a posteriori with uniform prior

- Uniform prior for Beta $\bar{\alpha} = 1$ $\bar{\beta} = 1$

$m^* =$



Posterior Estimates of Cells

- Maximum a posteriori (mode of Beta)

$$m^* = \operatorname{argmax}_m P(m \mid x_1, \dots, x_t, z_1, \dots, z_t) = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2}$$

- Expected value (unbiased)

$$m^* = \mathbb{E}[P(m \mid x_1, \dots, x_t, z_1, \dots, z_t)] = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}$$

- Maximum likelihood (revised)

- Maximum a posteriori with uniform prior

- Uniform prior for Beta $\bar{\alpha} = 1$ $\bar{\beta} = 1$

$$m^* = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2} =$$



Posterior Estimates of Cells

- Maximum a posteriori (mode of Beta)

$$m^* = \operatorname{argmax}_m P(m \mid x_1, \dots, x_t, z_1, \dots, z_t) = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2}$$

- Expected value (unbiased)

$$m^* = \mathbb{E}[P(m \mid x_1, \dots, x_t, z_1, \dots, z_t)] = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}$$

- Maximum likelihood (revised)

- Maximum a posteriori with uniform prior

- Uniform prior for Beta $\bar{\alpha} = 1$ $\bar{\beta} = 1$

$$m^* = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2} = \frac{\bar{\alpha} + \alpha_j - 1}{\bar{\alpha} + \alpha_j + \bar{\beta} + \beta_j - 2} =$$



Posterior Estimates of Cells

- Maximum a posteriori (mode of Beta)

$$m^* = \operatorname{argmax}_m P(m \mid x_1, \dots, x_t, z_1, \dots, z_t) = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2}$$

- Expected value (unbiased)

$$m^* = \mathbb{E}[P(m \mid x_1, \dots, x_t, z_1, \dots, z_t)] = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}$$

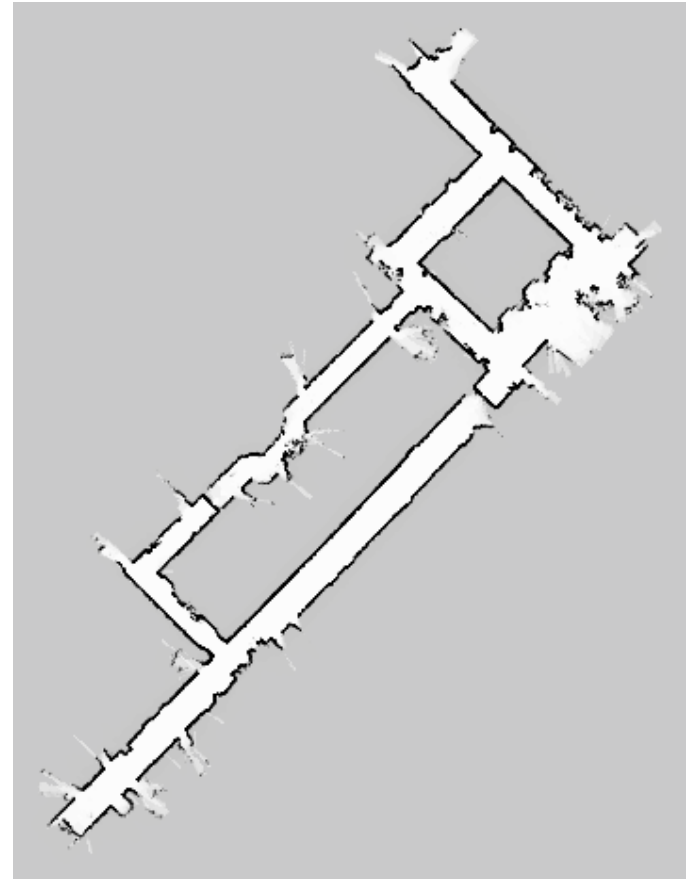
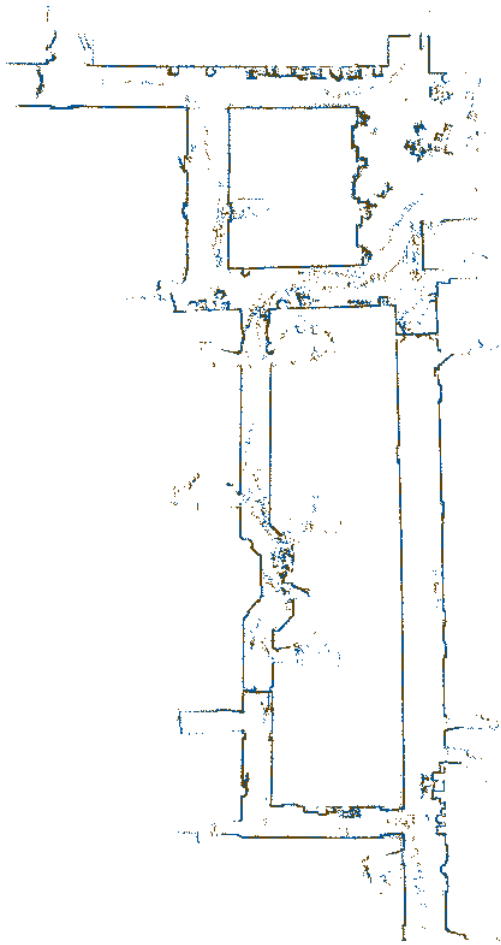
- Maximum likelihood (revised)

- Maximum a posteriori with uniform prior

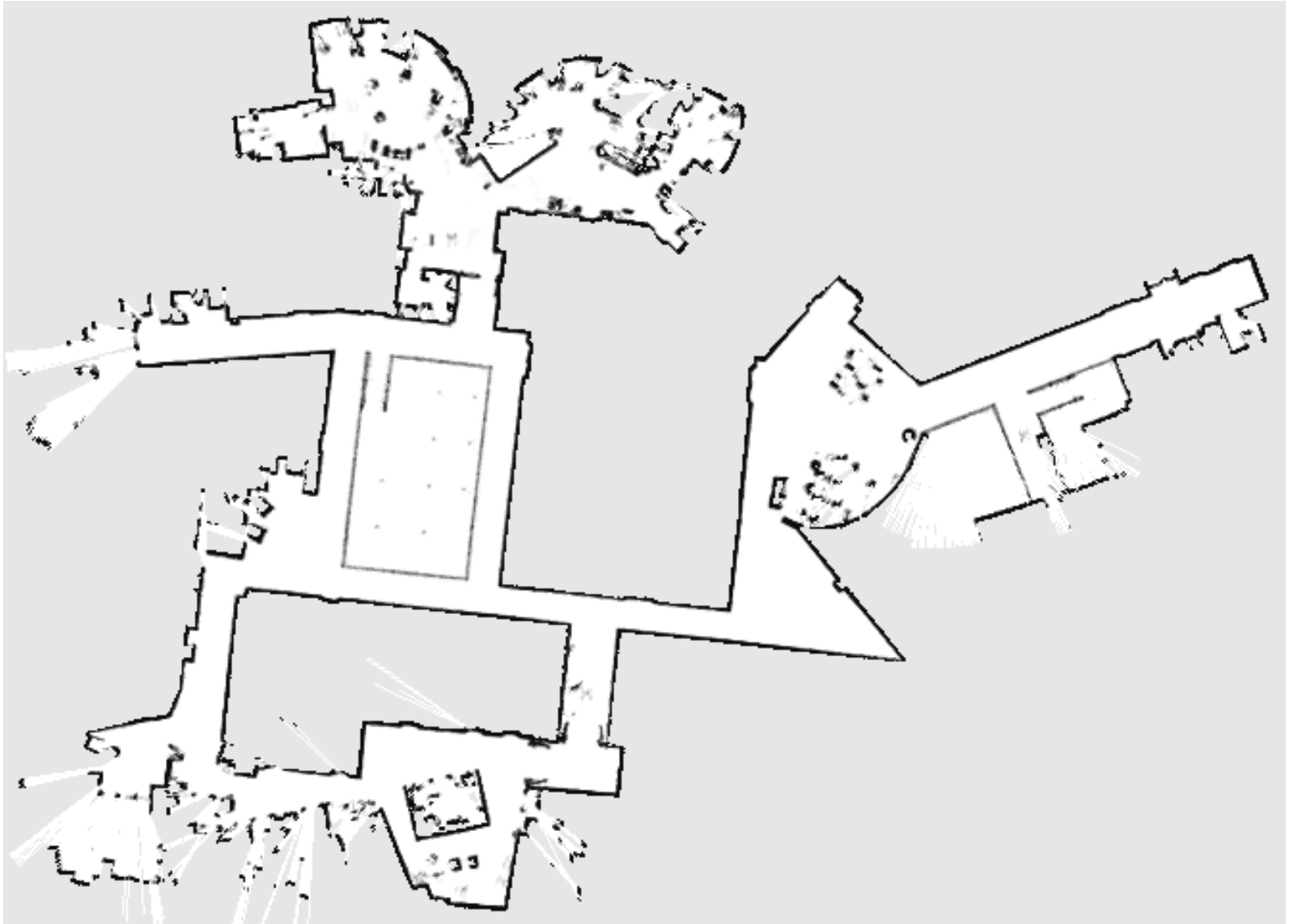
- Uniform prior for Beta $\bar{\alpha} = 1$ $\bar{\beta} = 1$

$$m^* = \frac{\hat{\alpha} - 1}{\hat{\alpha} + \hat{\beta} - 2} = \frac{\bar{\alpha} + \alpha_j - 1}{\bar{\alpha} + \alpha_j + \bar{\beta} + \beta_j - 2} = \frac{\alpha_j}{\alpha_j + \beta_j}$$

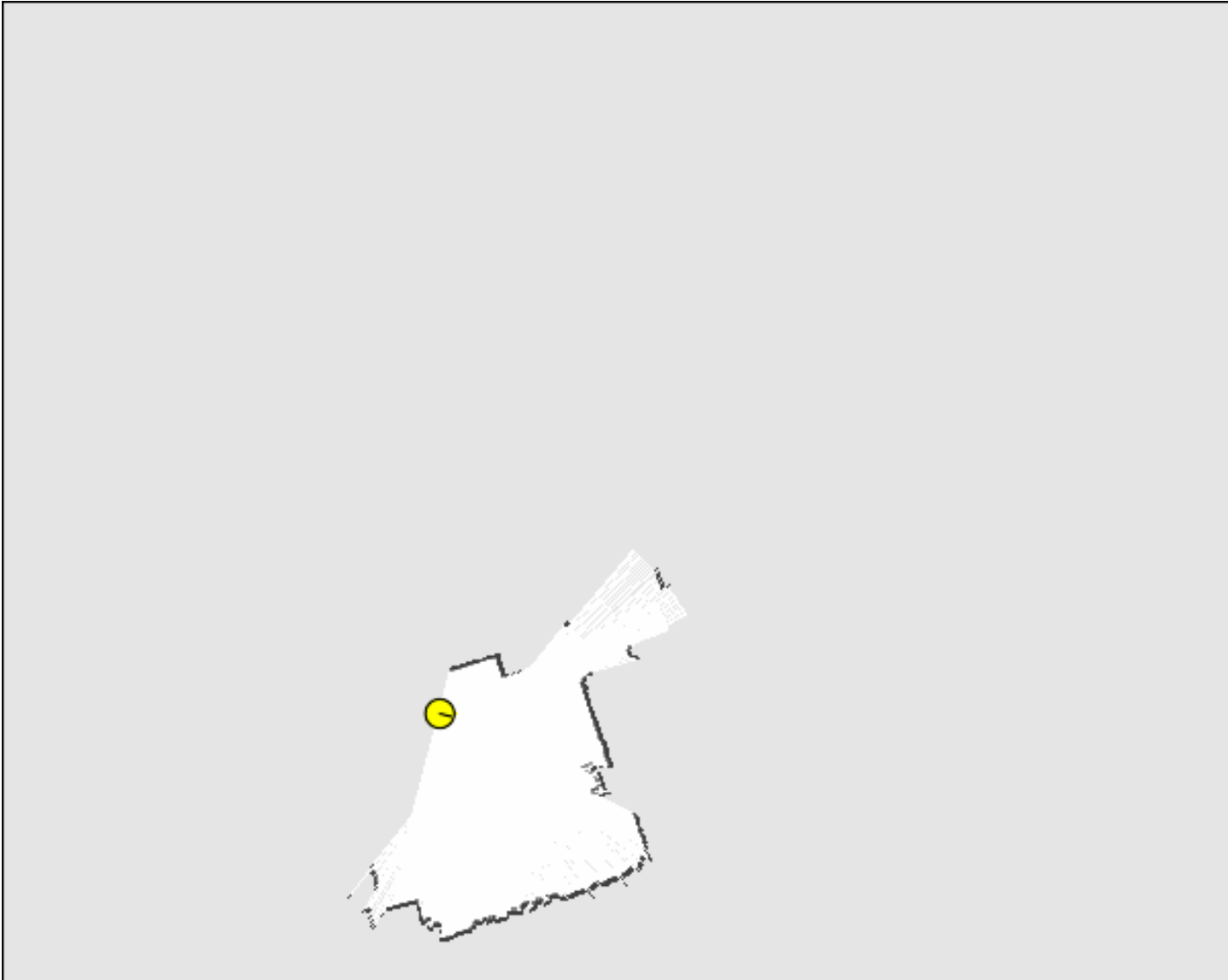
Occupancy Grids From Laser Scans to Maps



Example: MIT CSAIL 3rd Floor



Uni Freiburg Building 106



Occupancy Grid Map Summary

- Occupancy grid maps discretize the space into independent cells
- Each cell is a binary random variable estimating if the cell is occupied
- Static state binary Bayes filter per cell
- Mapping with known poses is easy
- Log odds model is fast to compute
- No need for predefined features

Literature

Static state binary Bayes filter

- Thrun et al.: “Probabilistic Robotics”, Chapter 4.2

Occupancy Grid Mapping

- Thrun et al.: “Probabilistic Robotics”, Chapter 9.1+9.2

Slide Information

- These slides have been created by Cyrill Stachniss as part of the robot mapping course taught in 2012/13 and 2013/14. I created this set of slides partially extending existing material of Edwin Olson, Pratik Agarwal, and myself.
- I tried to acknowledge all people that contributed image or video material. In case I missed something, please let me know. If you adapt this course material, please make sure you keep the acknowledgements.
- Feel free to use and change the slides. If you use them, I would appreciate an acknowledgement as well. To satisfy my own curiosity, I appreciate a short email notice in case you use the material in your course.
- My video recordings are available through YouTube:
http://www.youtube.com/playlist?list=PLgnQpQtFTOGQrZ4O5QzbIHgI3b1JHimN_&feature=g-list