

# Robot Mapping

## EKF SLAM

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# Simultaneous Localization and Mapping (SLAM)

- Building a map and locating the robot in the map at the same time
- Chicken-or-egg problem



# Definition of the SLAM Problem

## Given

- The robot's controls

$$u_{1:T} = \{u_1, u_2, u_3, \dots, u_T\}$$

- Observations

$$z_{1:T} = \{z_1, z_2, z_3, \dots, z_T\}$$

## Wanted

- Map of the environment

$$m$$

- Path of the robot

$$x_{0:T} = \{x_0, x_1, x_2, \dots, x_T\}$$

# Three Main Paradigms

Kalman  
filter

Particle  
filter

Graph-  
based

# Bayes Filter

- Recursive filter with prediction and correction step

- **Prediction**

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

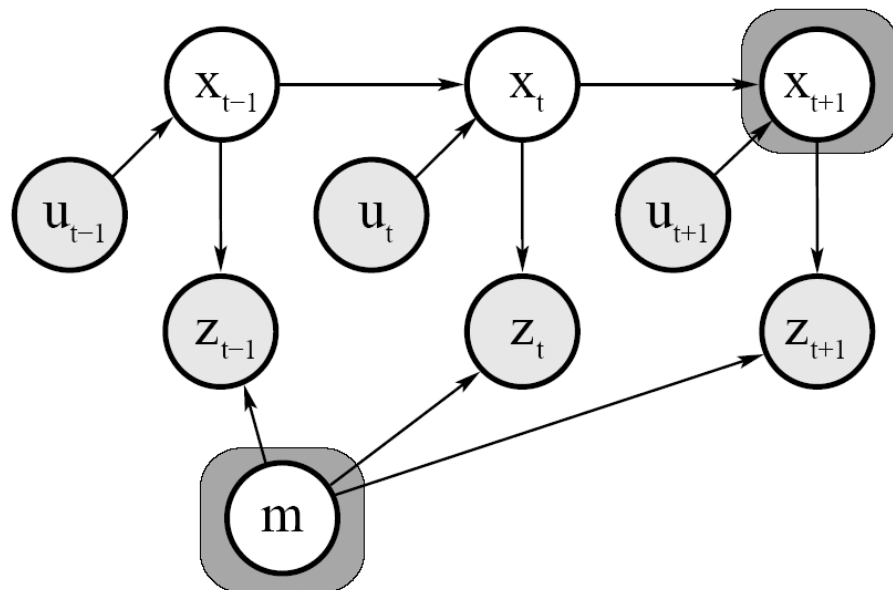
- **Correction**

$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

# EKF for Online SLAM

- We consider here the Kalman filter as a solution to the online SLAM problem

$$p(x_t, m \mid z_{1:t}, u_{1:t})$$



# Extended Kalman Filter Algorithm

- 1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
- 2:  $\bar{\mu}_t = g(u_t, \mu_{t-1})$
- 3:  $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$
- 4:  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$
- 5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
- 6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$
- 7: *return*  $\mu_t, \Sigma_t$

# EKF SLAM

- Application of the EKF to SLAM
- Estimate robot's pose and locations of landmarks in the environment
- Assumption: known correspondences
- State space (for the 2D plane) is

$$x_t = \left( \underbrace{x, y, \theta}_{\text{robot's pose}}, \underbrace{m_{1,x}, m_{1,y}}_{\text{landmark 1}}, \dots, \underbrace{m_{n,x}, m_{n,y}}_{\text{landmark n}} \right)^T$$



# EKF SLAM: State Representation

- Map with  $n$  landmarks:  $(3+2n)$ -dimensional Gaussian
- Belief is represented by

$$\underbrace{\begin{pmatrix} x \\ y \\ \theta \\ m_{1,x} \\ m_{1,y} \\ \vdots \\ m_{n,x} \\ m_{n,y} \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xm_{1,x}} & \sigma_{xm_{1,y}} & \dots & \sigma_{xm_{n,x}} & \sigma_{xm_{n,y}} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{ym_{1,x}} & \sigma_{ym_{1,y}} & \dots & \sigma_{m_{n,x}} & \sigma_{m_{n,y}} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta m_{1,x}} & \sigma_{\theta m_{1,y}} & \dots & \sigma_{\theta m_{n,x}} & \sigma_{\theta m_{n,y}} \\ \sigma_{m_{1,x}x} & \sigma_{m_{1,x}y} & \sigma_{\theta} & \sigma_{m_{1,x}m_{1,x}} & \sigma_{m_{1,x}m_{1,y}} & \dots & \sigma_{m_{1,x}m_{n,x}} & \sigma_{m_{1,x}m_{n,y}} \\ \sigma_{m_{1,y}x} & \sigma_{m_{1,y}y} & \sigma_{\theta} & \sigma_{m_{1,y}m_{1,x}} & \sigma_{m_{1,y}m_{1,y}} & \dots & \sigma_{m_{1,y}m_{n,x}} & \sigma_{m_{1,y}m_{n,y}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{m_{n,x}x} & \sigma_{m_{n,x}y} & \sigma_{\theta} & \sigma_{m_{n,x}m_{1,x}} & \sigma_{m_{n,x}m_{1,y}} & \dots & \sigma_{m_{n,x}m_{n,x}} & \sigma_{m_{n,x}m_{n,y}} \\ \sigma_{m_{n,y}x} & \sigma_{m_{n,y}y} & \sigma_{\theta} & \sigma_{m_{n,y}m_{1,x}} & \sigma_{m_{n,y}m_{1,y}} & \dots & \sigma_{m_{n,y}m_{n,x}} & \sigma_{m_{n,y}m_{n,y}} \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: State Representation

- More compactly

$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \sum x_R x_R & \sum x_R m_1 & \cdots & \sum x_R m_n \\ \sum m_1 x_R & \sum m_1 m_1 & \cdots & \sum m_1 m_n \\ \vdots & \vdots & \ddots & \vdots \\ \sum m_n x_R & \sum m_n m_1 & \cdots & \sum m_n m_n \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: State Representation

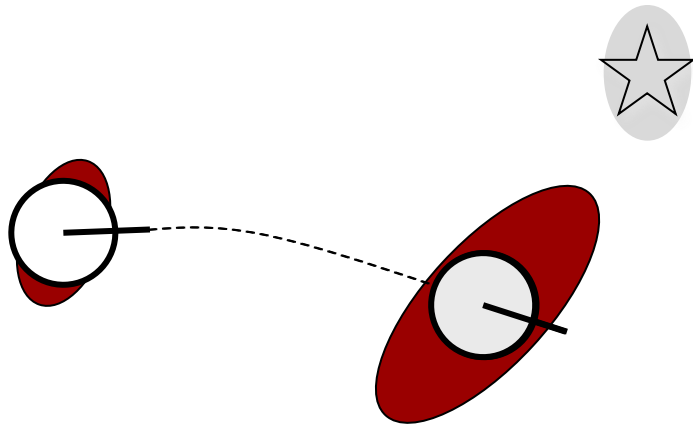
- Even more compactly (note:  $x_R \rightarrow x$ )

$$\underbrace{\begin{pmatrix} x \\ m \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: Filter Cycle

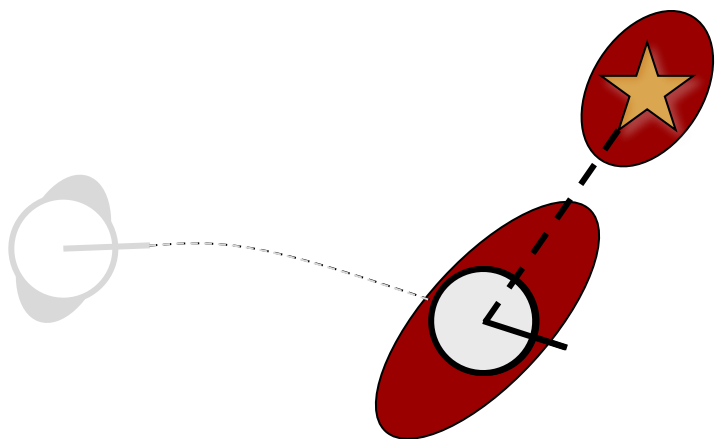
1. State prediction
2. Measurement prediction
3. Measurement
4. Data association
5. Update

# EKF SLAM: State Prediction



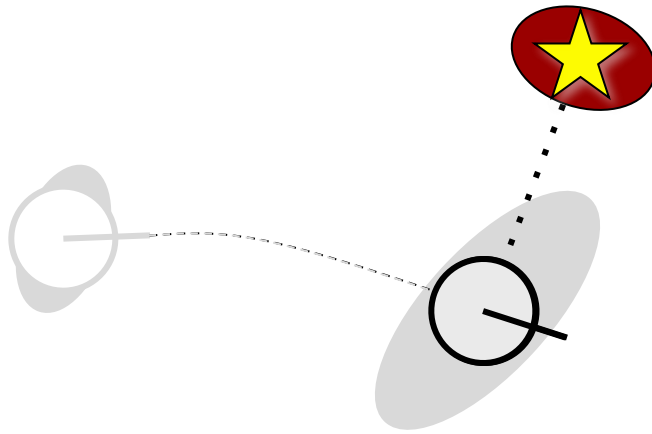
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: Measurement Prediction



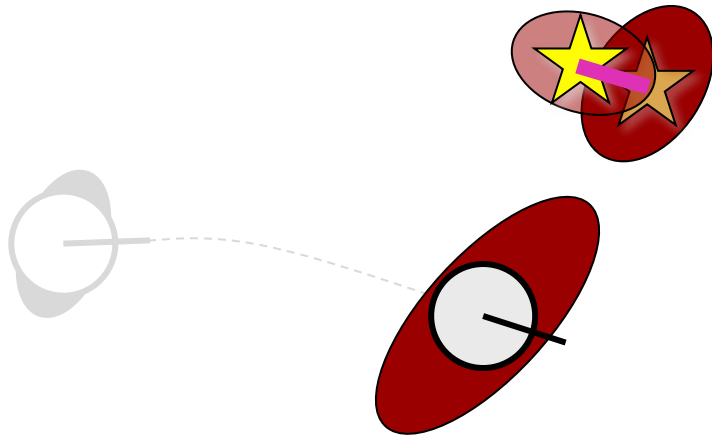
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: Obtained Measurement



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

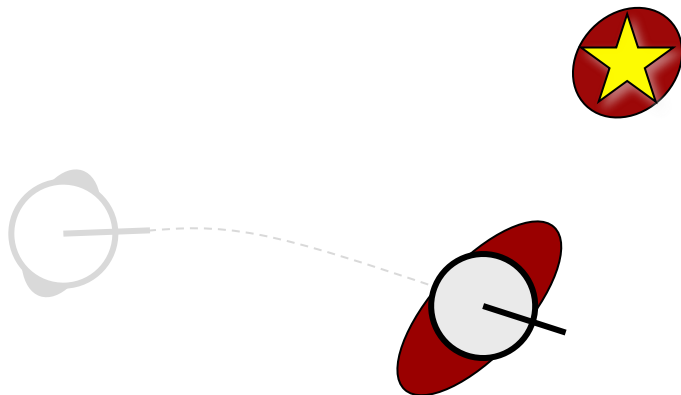
# EKF SLAM: Data Association and Difference Between $h(x)$ and $z$



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$



# EKF SLAM: Update Step



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

# EKF SLAM: Concrete Example

## Setup

- Robot moves in the 2D plane
- Velocity-based motion model
- Robot observes point landmarks
- Range-bearing sensor
- Known data association
- Known number of landmarks

# Initialization

- Robot starts in its own reference frame (all landmarks unknown)
- $2N+3$  dimensions

$$\mu_0 = (0 \ 0 \ 0 \ \dots \ 0)^T$$
$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \infty & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \infty \end{pmatrix}$$

# Extended Kalman Filter Algorithm

1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):

2:  $\bar{\mu}_t = g(u_t, \mu_{t-1})$

3:  $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$

4:  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$

7: *return*  $\mu_t, \Sigma_t$

# Prediction Step (Motion)

- Goal: Update state space based on the robot's motion
- Robot motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g_{x,y,\theta}(u_t, (x,y,\theta)^T)}$$

- How to map that to the  $2N+3$  dim space?

# Update the State Space


- From the motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

- to the  $2N+3$  dimensional space

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \\ \vdots \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 \dots 0 \\ 0 & 0 & 1 & \underbrace{0 \dots 0}_{2N \text{ cols}} \end{pmatrix}^T}_{F_x^T} \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g(u_t, x_t)}$$

# Extended Kalman Filter Algorithm

- 1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
- 2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **DONE**
- 3:  $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$   

- 4:  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$
- 5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
- 6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$
- 7: *return*  $\mu_t, \Sigma_t$

# Update Covariance

- The function  $g$  only affects the robot's motion and not the landmarks

Jacobian of the motion (3x3)

$$G_t = \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix}$$

Identity (2N x 2N)



# Jacobian of the Motion

$$G_t^x = \frac{\partial}{\partial(x, y, \theta)^T} \left[ \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \right]$$

# Jacobian of the Motion

$$\begin{aligned} G_t^x &= \frac{\partial}{\partial(x, y, \theta)^T} \left[ \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \right] \\ &= I + \frac{\partial}{\partial(x, y, \theta)^T} \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \end{aligned}$$

# Jacobian of the Motion

$$\begin{aligned} G_t^x &= \frac{\partial}{\partial(x, y, \theta)^T} \left[ \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \right] \\ &= I + \frac{\partial}{\partial(x, y, \theta)^T} \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \\ &= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \theta + \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$


# Jacobian of the Motion


$$\begin{aligned}
 G_t^x &= \frac{\partial}{\partial(x, y, \theta)^T} \left[ \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \right] \\
 &= I + \frac{\partial}{\partial(x, y, \theta)^T} \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \\
 &= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \theta + \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t} \cos \theta + \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

# This Leads to the Update

1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):

2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **Apply & DONE**

3:   $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$


$$\begin{aligned}\bar{\Sigma}_t &= G_t \Sigma_{t-1} G_t^T + R_t \\ &= \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix} \begin{pmatrix} (G_t^x)^T & 0 \\ 0 & I \end{pmatrix} + R_t \\ &= \begin{pmatrix} G_t^x \Sigma_{xx} (G_t^x)^T & G_t^x \Sigma_{xm} \\ (G_t^x \Sigma_{xm})^T & \Sigma_{mm} \end{pmatrix} + R_t\end{aligned}$$

# Extended Kalman Filter Algorithm

- 1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
- 2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **DONE**
- 3:  ~~$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$~~  **DONE**
- 4:  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$
- 5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
- 6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$
- 7: *return*  $\mu_t, \Sigma_t$

# EKF SLAM: Prediction Step

**EKF\_SLAM\_Prediction**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, R_t$ ):


$$2: F_x = \begin{pmatrix} 1 & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 \dots 0 \\ 0 & 0 & 1 & 0 \dots 0 \end{pmatrix}$$

$$3: \bar{\mu}_t = \mu_{t-1} + F_x^T \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

$$4: G_t = I + F_x^T \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} + \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} F_x$$

$$5: \bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + \underbrace{F_x^T R_t^x F_x}_{R_t}$$

# Extended Kalman Filter Algorithm

- 1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
- 2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **DONE**
- 3:  ~~$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$~~  **Apply & DONE**
- 4:  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$
- 5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
- 6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$
- 7: *return*  $\mu_t, \Sigma_t$



# EKF SLAM: Correction Step

- Known data association
- $c_t^i = j$ :  $i$ -th measurement at time  $t$  observes the landmark with index  $j$
- Initialize landmark if unobserved
- Compute the expected observation
- Compute the Jacobian of  $h$
- Proceed with computing the Kalman gain

# Range-Bearing Observation

- Range-Bearing observation  $z_t^i = (r_t^i, \phi_t^i)^T$
- If landmark has not been observed

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

observed  
location of  
landmark j

estimated  
robot's  
location

relative  
measurement

# Expected Observation

- Compute expected observation according to the current estimate

$$\begin{aligned}\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \\ &= h(\bar{\mu}_t)\end{aligned}$$

# Jacobian for the Observation

- Based on 
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
$$q = \delta^T \delta$$
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

- Compute the Jacobian

$${}^{\text{low}}H_t^i = \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t}$$

low-dim space  $(x, y, \theta, m_{j,x}, m_{j,y})$

# Jacobian for the Observation

- Based on 
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
$$q = \delta^T \delta$$
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

- Compute the Jacobian

$$\begin{aligned} \text{low } H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \begin{pmatrix} \frac{\partial \sqrt{q}}{\partial x} & \frac{\partial \sqrt{q}}{\partial y} & \dots \\ \frac{\partial \text{atan2}(\dots)}{\partial x} & \frac{\partial \text{atan2}(\dots)}{\partial y} & \dots \end{pmatrix} \end{aligned}$$

# The First Component

- Based on 
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
$$q = \delta^T \delta$$
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

- We obtain (by applying the chain rule)

$$\begin{aligned} \frac{\partial \sqrt{q}}{\partial x} &= \frac{1}{2} \frac{1}{\sqrt{q}} 2 \delta_x (-1) \\ &= \frac{1}{q} (-\sqrt{q} \delta_x) \end{aligned}$$

# Jacobian for the Observation

- Based on 
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
$$q = \delta^T \delta$$
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

- Compute the Jacobian

$$\begin{aligned} \text{low } H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix} \end{aligned}$$

# Jacobian for the Observation

- Use the computed Jacobian

$$\text{low } H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}$$

- map it to the high dimensional space

$$H_t^i = \text{low } H_t^i F_{x,j}$$

$\downarrow$   
 $F_{x,j} =$

$\left( \begin{array}{ccccccc} 1 & 0 & 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 0 & 1 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 0 & 0 & 0 \dots 0 & 1 & 0 & 0 \dots 0 \\ 0 & 0 & 0 & \underbrace{0 \dots 0}_{2j-2} & 0 & 1 & \underbrace{0 \dots 0}_{2N-2j} \end{array} \right)$




# Next Steps as Specified...

1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):

2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **DONE**

3:  ~~$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$~~  **DONE**


4:   $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

5:  $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6:  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$

7: *return*  $\mu_t, \Sigma_t$

# Extended Kalman Filter Algorithm

- 1: **Extended\_Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
- 2:  ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~  **DONE**
- 3:  ~~$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$~~  **DONE**
- 4:  ~~$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$~~  **Apply & DONE**
- 5:  ~~$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$~~  **Apply & DONE**
- 6:  ~~$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$~~  **Apply & DONE**
- 7:  **return**  $\mu_t, \Sigma_t$

# EKF SLAM – Correction (1/2)

## EKF\_SLAM\_Correction

```
6:  $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{pmatrix}$ 
7: for all observed features  $z_t^i = (r_t^i, \phi_t^i)^T$  do
8:    $j = c_t^i$ 
9:   if landmark  $j$  never seen before
10:     $\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$ 
11:   endif
12:    $\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$ 
13:    $q = \delta^T \delta$ 
14:    $\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$ 
```

# EKF SLAM – Correction (2/2)

```

15:    $F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 0 & 1 & 0 \dots 0 & 0 & 0 & 0 \dots 0 \\ 0 & 0 & 0 & 0 \dots 0 & 1 & 0 & 0 \dots 0 \\ 0 & 0 & 0 & \underbrace{0 \dots 0}_{2j-2} & 0 & 1 & \underbrace{0 \dots 0}_{2N-2j} \end{pmatrix}$ 
16:    $H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & +\delta_x \end{pmatrix} F_{x,j}$ 
17:    $K_t^i = \bar{\Sigma}_t H_t^{iT} (H_t^i \bar{\Sigma}_t H_t^{iT} + Q_t)^{-1}$ 
18:    $\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$ 
19:    $\bar{\Sigma}_t = (I - K_t^i H_t^i) \bar{\Sigma}_t$ 
20:   endfor
21:    $\mu_t = \bar{\mu}_t$ 
22:    $\Sigma_t = \bar{\Sigma}_t$ 
23:   return  $\mu_t, \Sigma_t$ 

```

# Implementation Notes

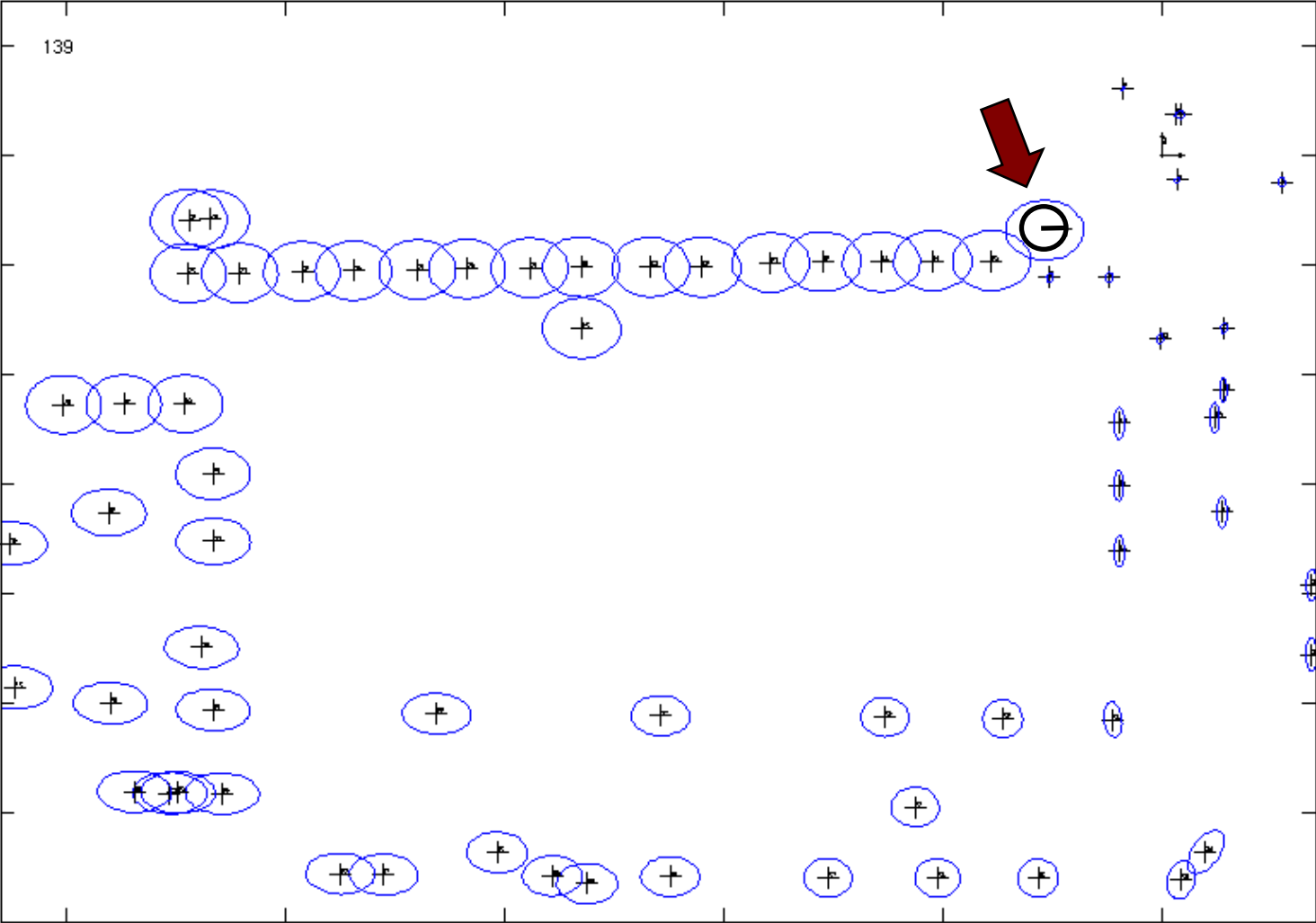
- Measurement update in a single step requires only one full belief update
- Always normalize the angular components
- You may not need to create the  $F$  matrices explicitly (e.g., in Octave)

**Done!**

# Loop Closing

- Loop closing means recognizing an already mapped area
- Data association under
  - high ambiguity
  - possible environment symmetries
- Uncertainties **collapse** after a loop closure (whether the closure was correct or not)

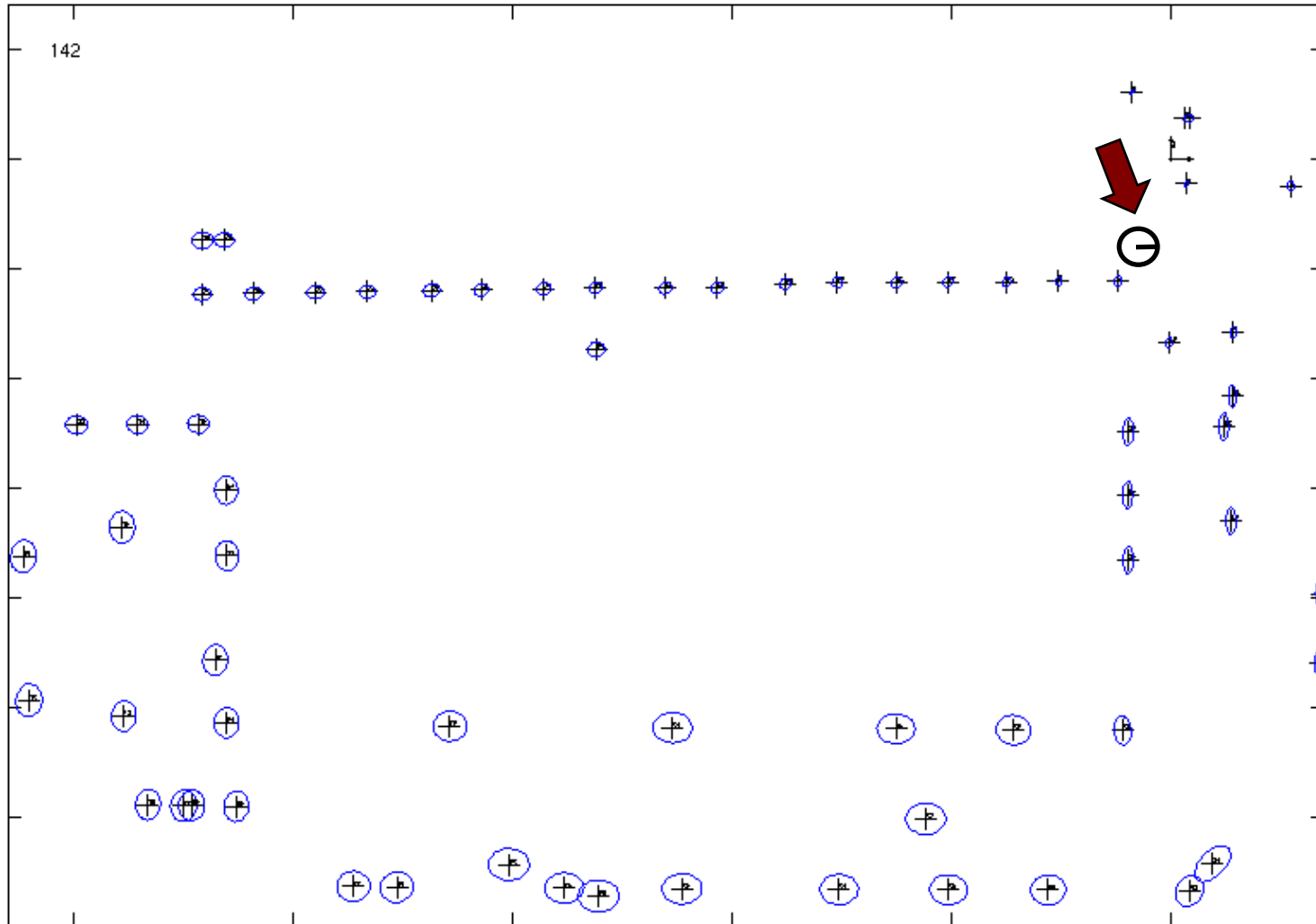
# Before the Loop Closure



Courtesy: K. Arras



# After the Loop Closure



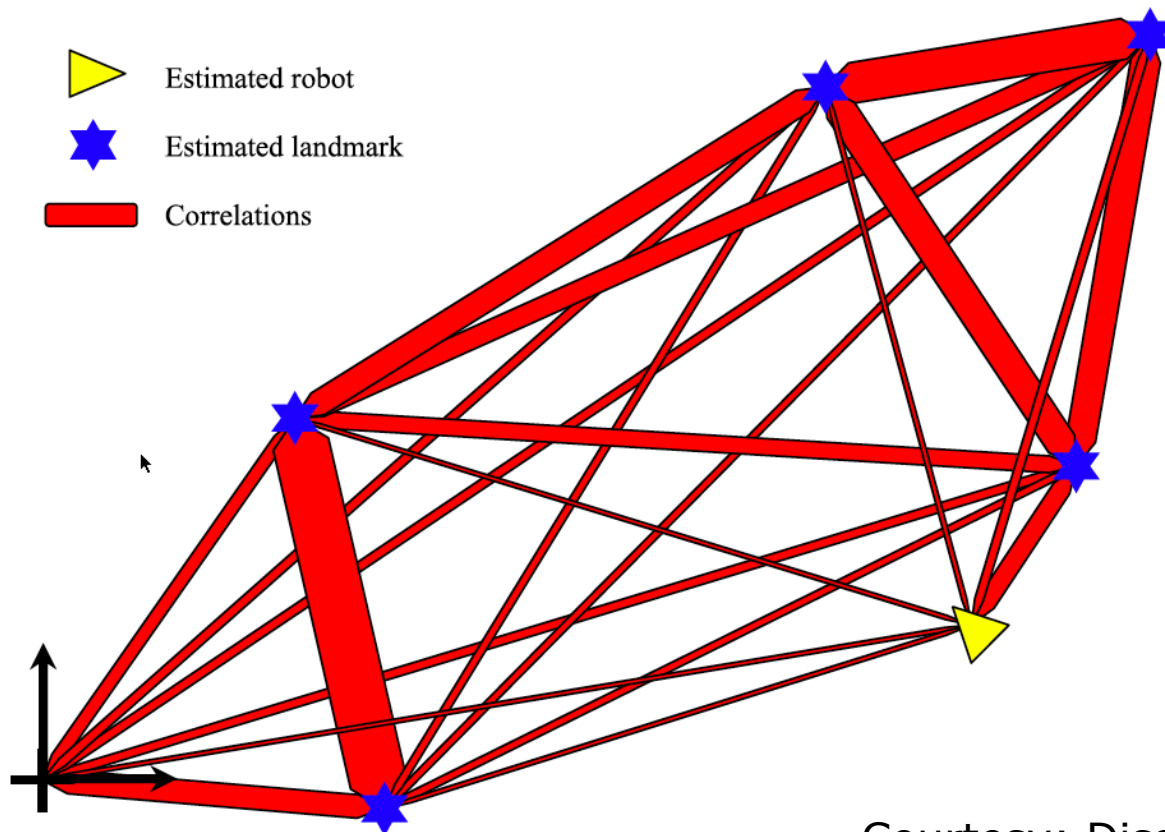
Courtesy: K. Arras

# Loop Closures in SLAM

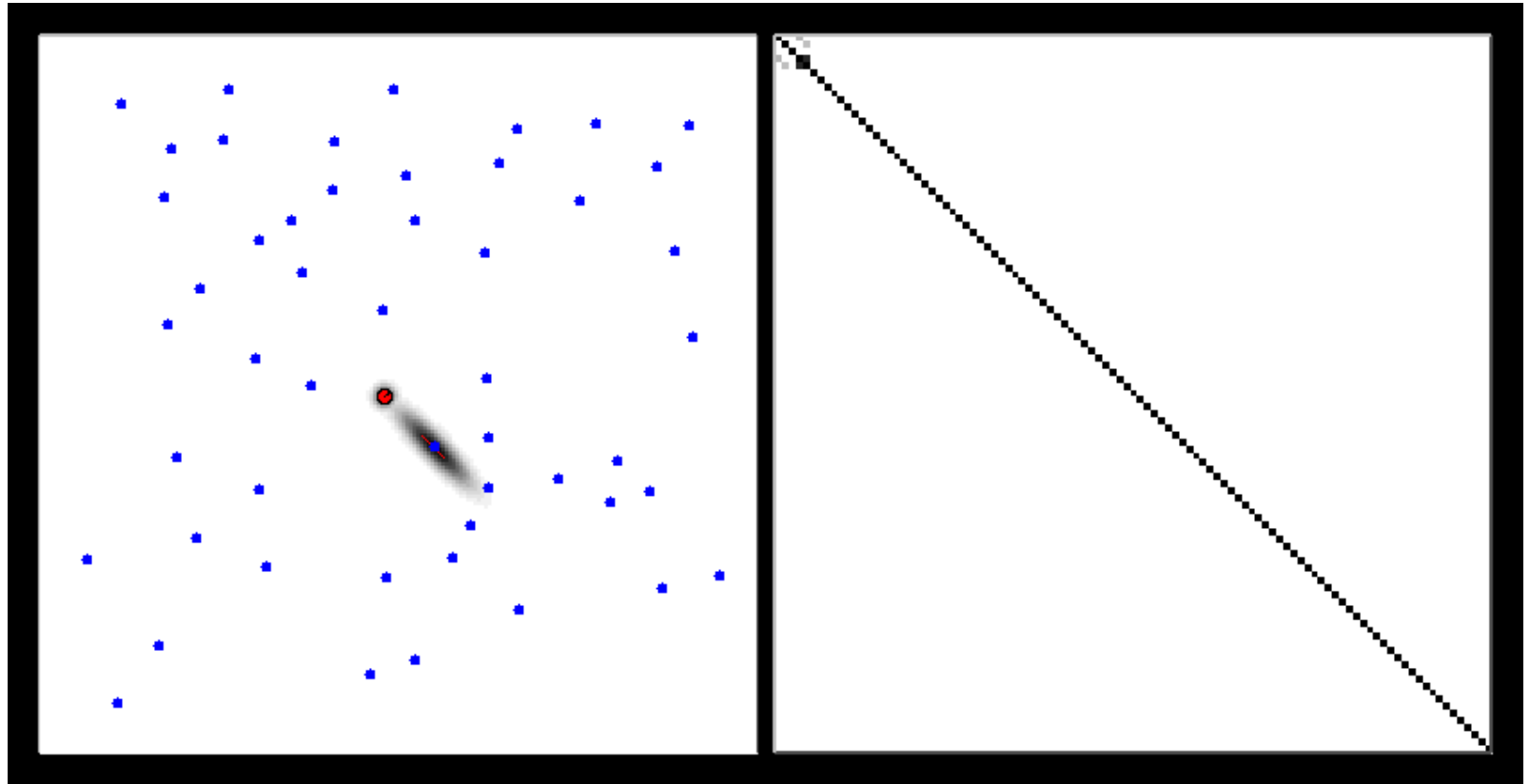
- Loop closing **reduces** the uncertainty in robot and landmark estimates
- This can be exploited when exploring an environment for the sake of better (e.g. more accurate) maps
- **Wrong loop closures lead to filter divergence**

# EKF SLAM Correlations

- In the limit, the landmark estimates become **fully correlated**



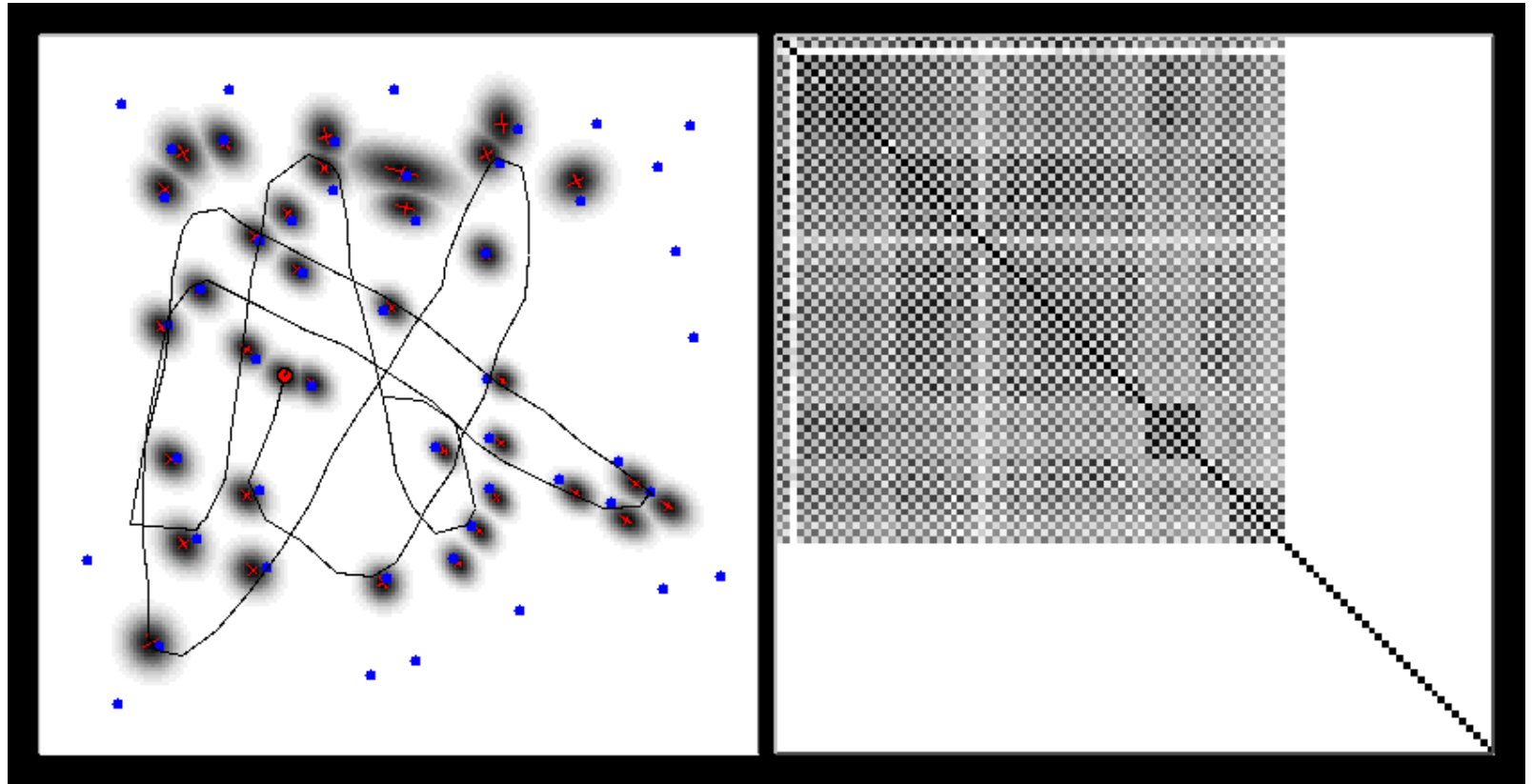
# EKF SLAM Correlations



Map

Correlation matrix

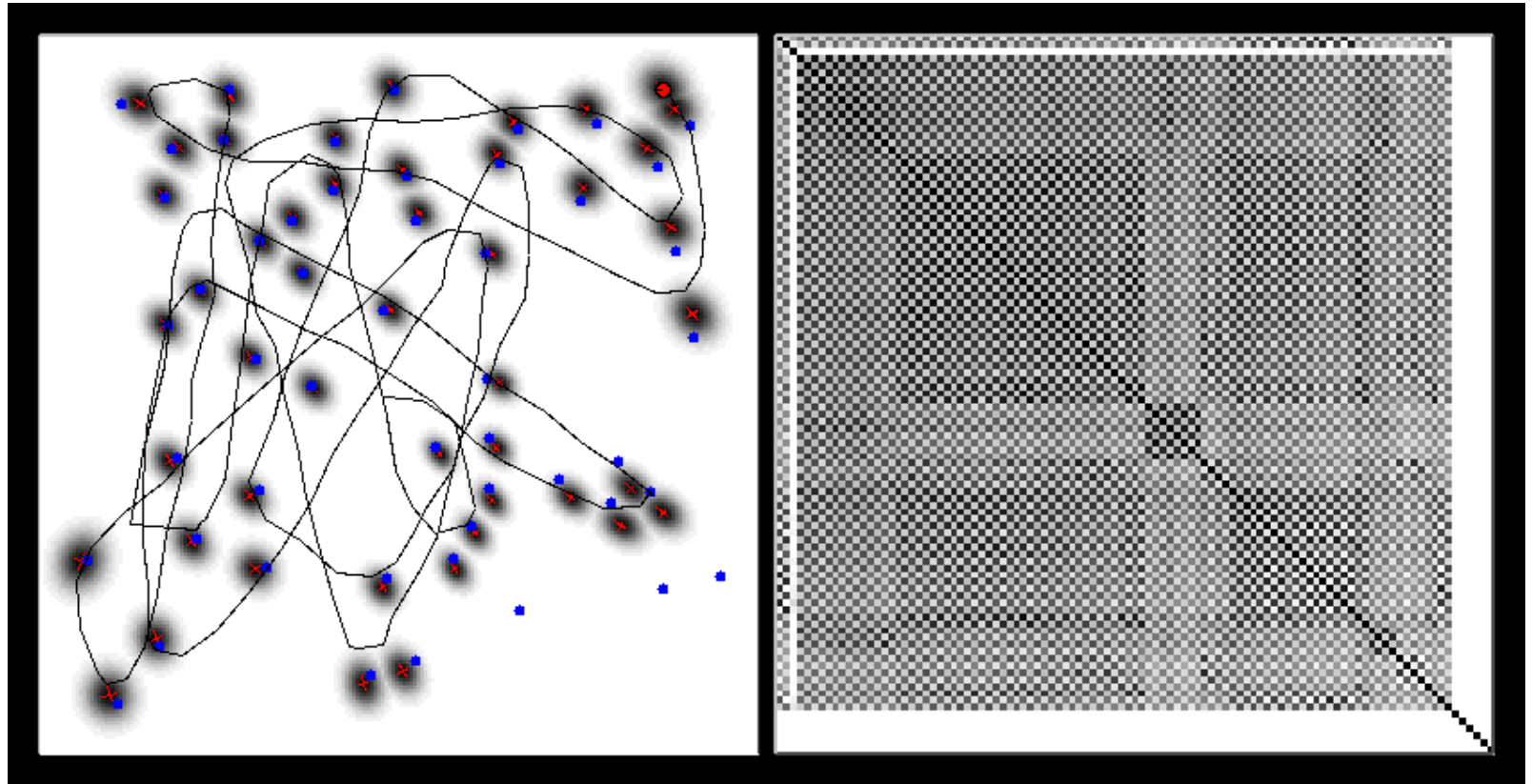
# EKF SLAM Correlations



Map

Correlation matrix

# EKF SLAM Correlations



Map

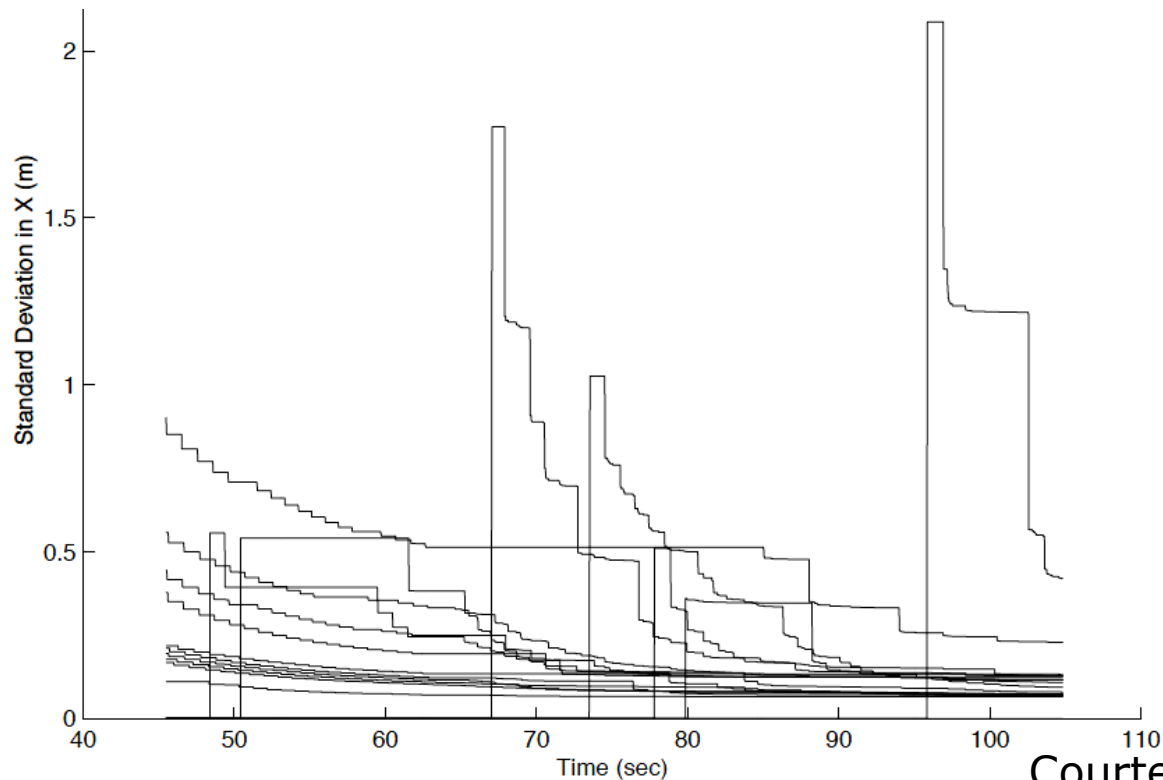
Correlation matrix

# EKF SLAM Correlations

- The correlation between the robot's pose and the landmarks **cannot** be ignored
- Assuming independence generates too optimistic estimates of the uncertainty

# EKF SLAM Uncertainties

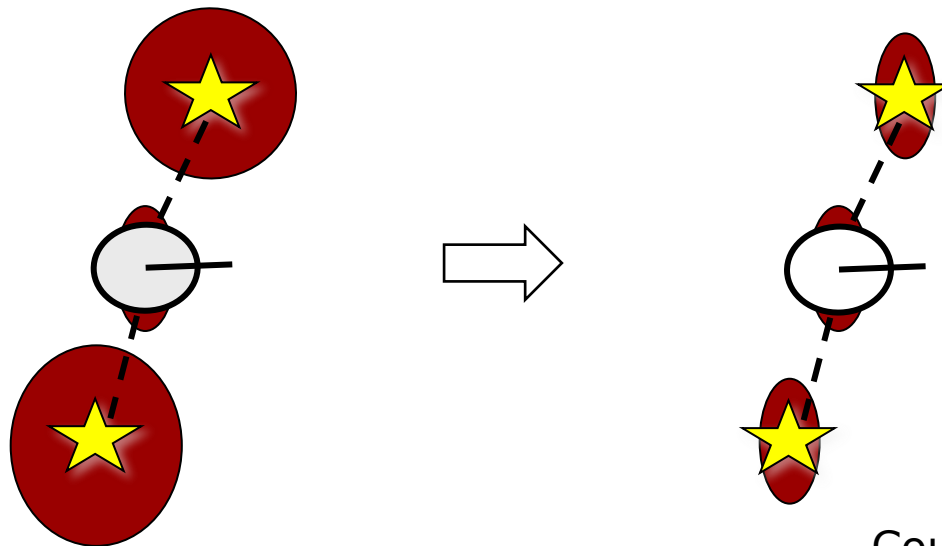
- The **determinant** of any sub-matrix of the map covariance matrix **decreases monotonically**
- New landmarks are initialized with **maximum uncertainty**





# EKF SLAM in the Limit

- In the limit, the covariance associated with any single landmark location estimate is determined only by the initial covariance in the vehicle location estimate.



# Example: Victoria Park Dataset



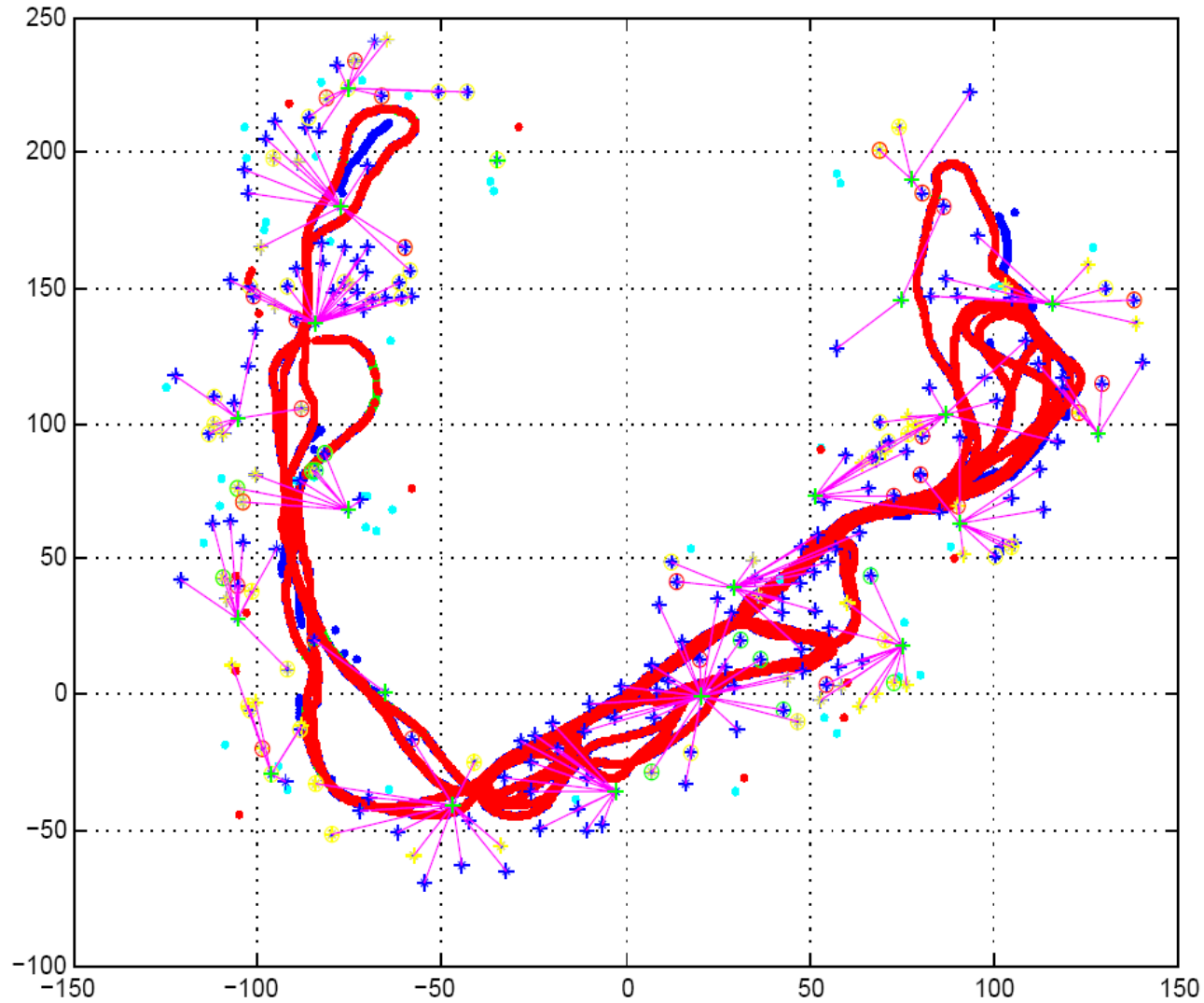
Courtesy: E. Nebot

# Victoria Park: Data Acquisition



Courtesy: E. Nebot

# Victoria Park: EKF Estimate



Courtesy: E. Nebot



# Victoria Park: Landmarks



Courtesy: E. Nebot

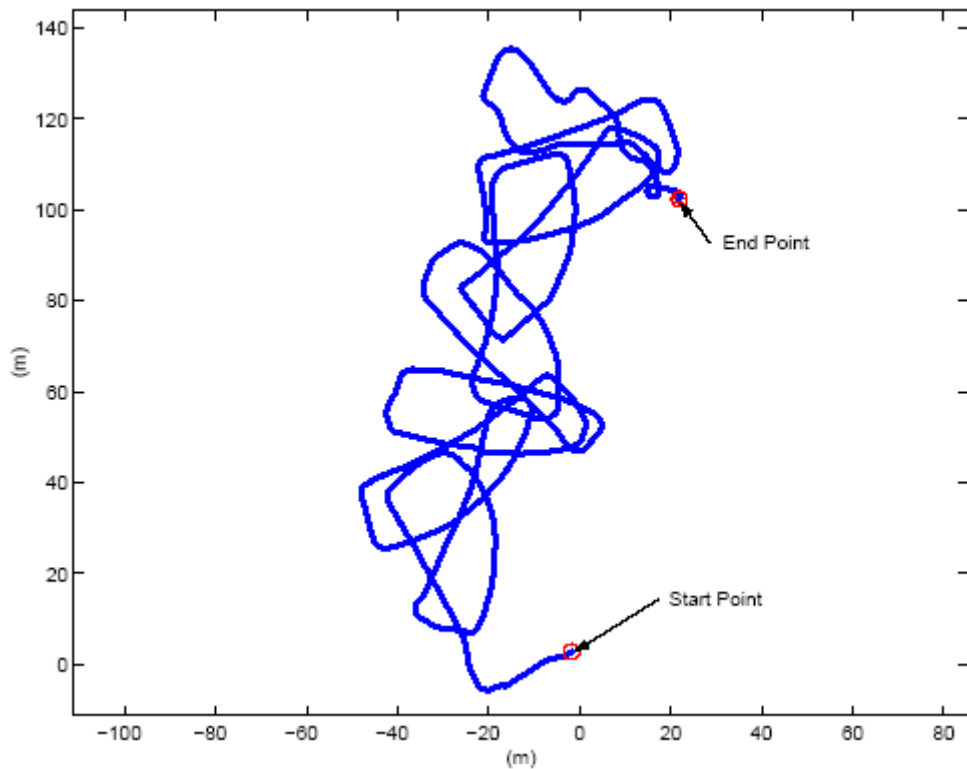
# Example: Tennis Court Dataset



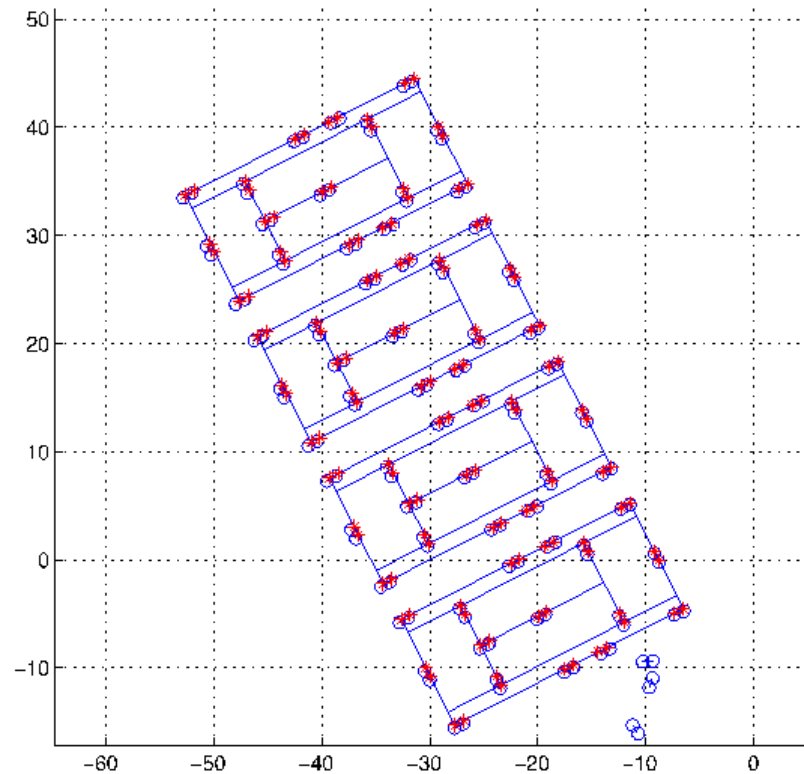
Courtesy: J. Leonard and M. Walter

# EKF SLAM on a Tennis Court

Odometry Profile of the Robot Locations



odometry



estimated trajectory

# EKF SLAM Complexity

- Cubic complexity depends only on the measurement dimensionality
- Cost per step: dominated by the number of landmarks:  $O(n^2)$
- Memory consumption:  $O(n^2)$
- The EKF becomes computationally intractable for large maps!



# EKF SLAM Summary

- The first SLAM solution
- Convergence proof for the linear Gaussian case
- Can diverge if non-linearities are large (and the reality is non-linear...)
- Can deal only with a single mode
- Successful in medium-scale scenes
- Approximations exist to reduce the computational complexity

# Literature

## EKF SLAM

- Thrun et al.: “Probabilistic Robotics”, Chapter 10

# Slide Information

- These slides have been created by Cyrill Stachniss as part of the robot mapping course taught in 2012/13 and 2013/14. I created this set of slides partially extending existing material of Edwin Olson, Pratik Agarwal, and myself.
- I tried to acknowledge all people that contributed image or video material. In case I missed something, please let me know. If you adapt this course material, please make sure you keep the acknowledgements.
- Feel free to use and change the slides. If you use them, I would appreciate an acknowledgement as well. To satisfy my own curiosity, I appreciate a short email notice in case you use the material in your course.
- My video recordings are available through YouTube:  
[http://www.youtube.com/playlist?list=PLgnQpQtFTOGQrZ4O5QzbIHgl3b1JHimN\\_&feature=g-list](http://www.youtube.com/playlist?list=PLgnQpQtFTOGQrZ4O5QzbIHgl3b1JHimN_&feature=g-list)