# Robot Mapping

# Least Squares SLAM Revisited & Hierarchical Approach to Least Squares SLAM

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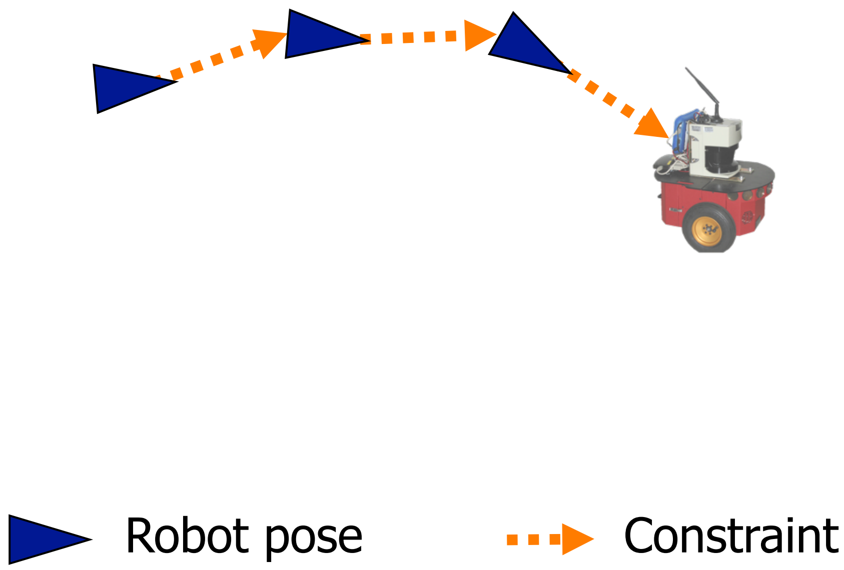
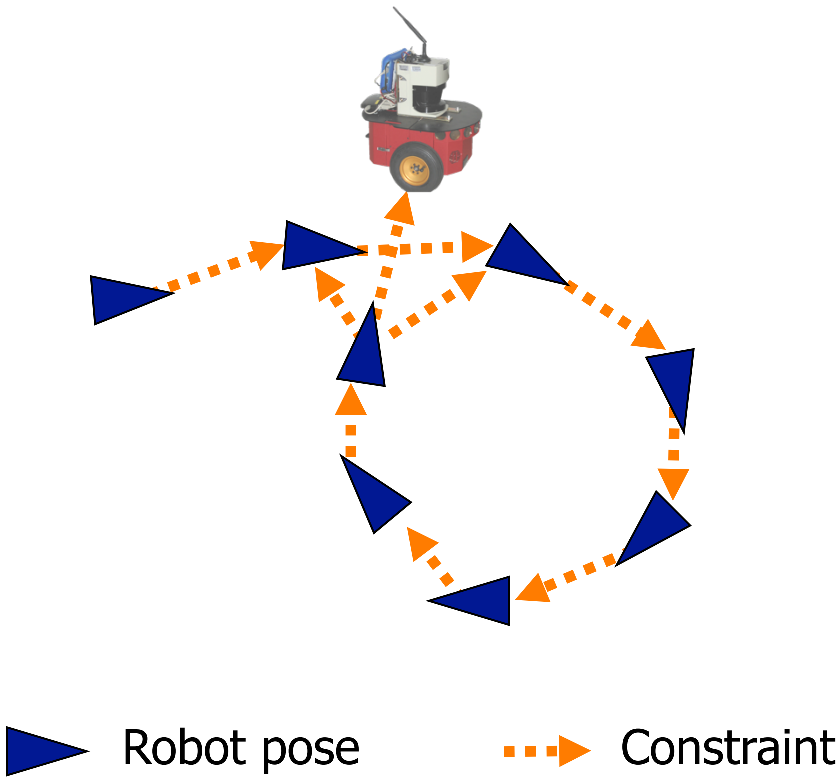
# Graph-Based SLAM

- Constraints connect the poses of the robot while it is moving
- Constraints are inherently uncertain



# Graph-Based SLAM

- Observing previously seen areas generates constraints between non-successive poses

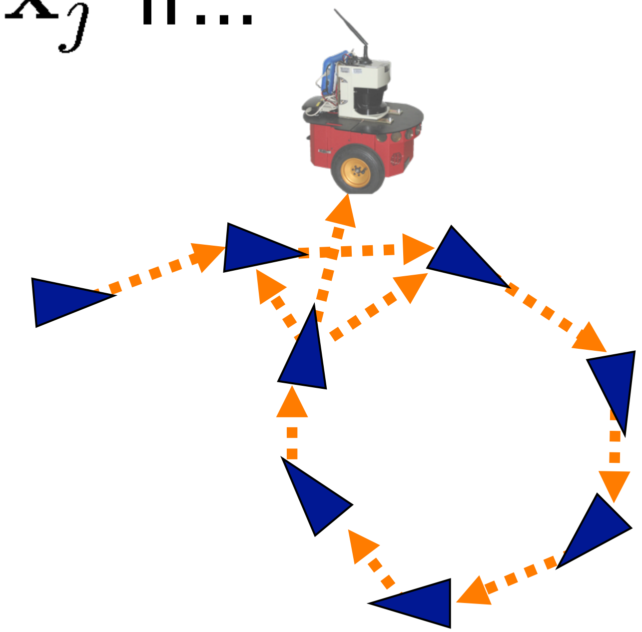


# Idea of Graph-Based SLAM

- Use a **graph** to represent the problem
- Every **node** in the graph corresponds to a pose of the robot during mapping
- Every **edge** between two nodes corresponds to a spatial constraint between them
- **Graph-Based SLAM:** Build the graph and find a node configuration that minimize the error introduced by the constraints

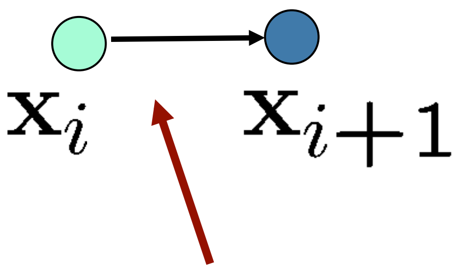
# The Graph

- It consists of n nodes $\mathbf{x} = \mathbf{x}_{1:n}$
- Each $\mathbf{x}_i$ is a 2D or 3D transformation (the pose of the robot at time $t_i$)
- A constraint/edge exists between the nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ if…



# Create an Edge If... (1)

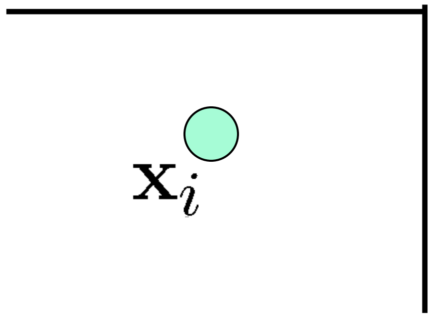
- ...the robot moves from $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$
- Edge corresponds to odometry

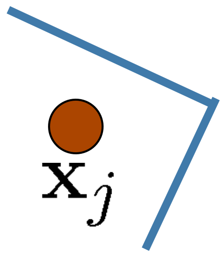



The edge represents the
**odometry** measurement

# Create an Edge If… (2)

- …the robot observes the same part of the environment from $\mathbf{x}_i$ and from $\mathbf{x}_j$



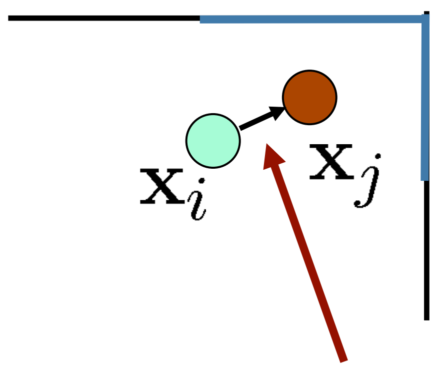




Measurement from $\mathbf{x}_i$

Measurement from $\mathbf{x}_j$

# Create an Edge If... (2)

- ...the robot observes the same part of the environment from $\mathbf{x}_i$ and from $\mathbf{x}_j$
- Construct a **virtual measurement** about the position of $\mathbf{x}_j$ seen from $\mathbf{x}_i$




Edge represents the position of $\mathbf{x}_j$ seen from $\mathbf{x}_i$ based on the **observation**

# Transformations

- Transformations can be expressed using **homogenous coordinates**
- Odometry-Based edge

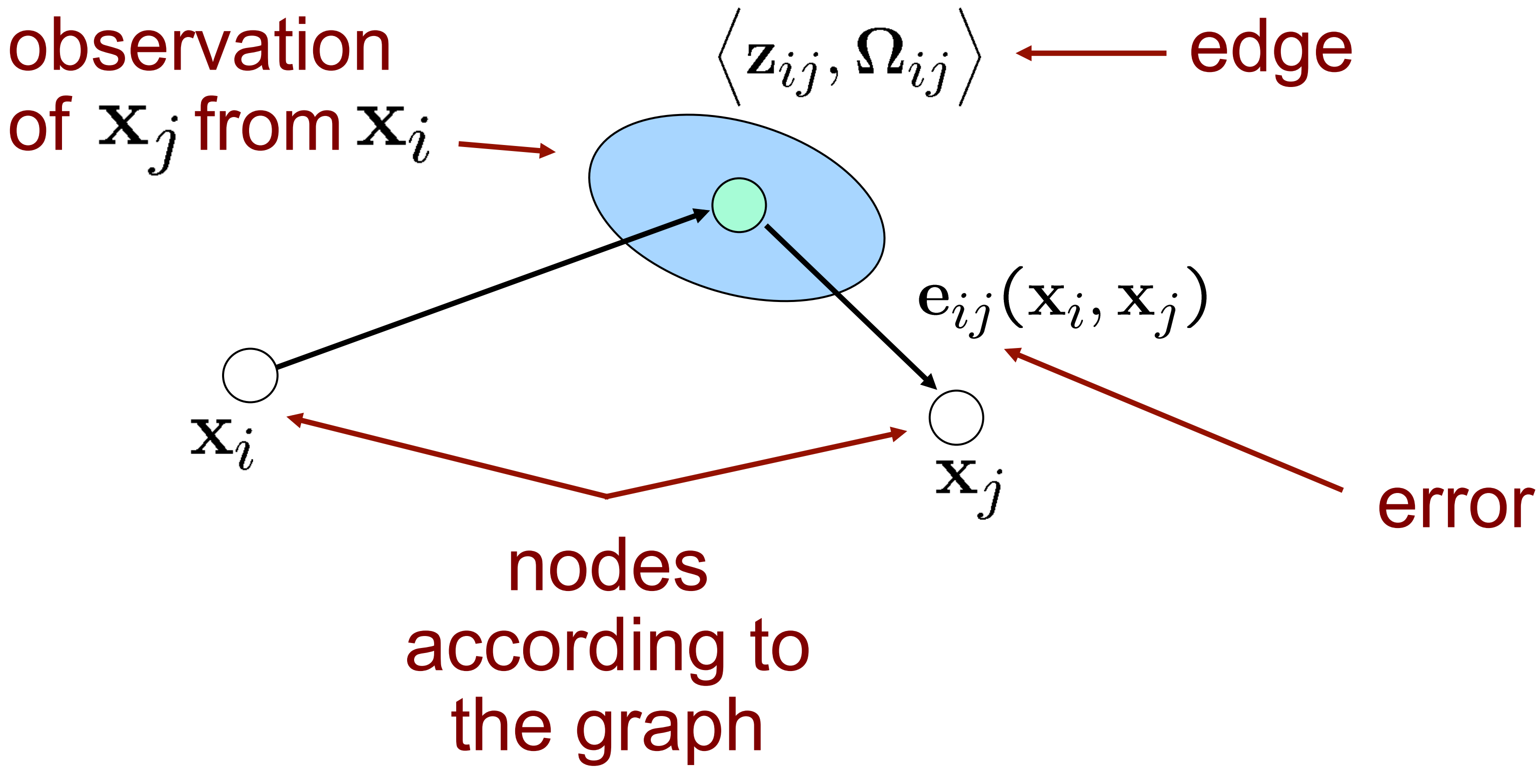
$$(\mathbf{X}_i^{-1}\mathbf{X}_{i+1})$$

- Observation-Based edge

$$(\mathbf{X}_i^{-1}\mathbf{X}_j)$$

How node i sees node j

# Pose Graph




- **Goal:** $\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \sum_{ij} \mathbf{e}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{e}_{ij}$

# The Error Function

- Error function for a single constraint

$$\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \text{t2v}(\underbrace{\mathbf{Z}_{ij}^{-1}}_{\text{measurement}}(\underbrace{\mathbf{X}_i^{-1}\mathbf{X}_j}_{\mathbf{x}_j \text{ referenced w.r.t. } \mathbf{x}_i}))$$

measurement

***x**$_j$* referenced w.r.t. ***x**$_i$*

- Error takes a value of zero if

$$\mathbf{Z}_{ij} = (\mathbf{X}_i^{-1}\mathbf{X}_j)$$

# Gauss-Newton: The Overall Error Minimization Procedure

- Define the error function
- Linearize the error function
- Compute its derivative
- Set the derivative to zero
- Solve the linear system
- Iterate this procedure until convergence

# Linearizing the Error Function

- We can approximate the error functions around an initial guess $\mathbf{x}$ via Taylor expansion

$$\mathbf{e}_{ij}(\mathbf{x} + \mathbf{\Delta x}) \simeq \mathbf{e}_{ij}(\mathbf{x}) + \mathbf{J}_{ij}\mathbf{\Delta x}$$

with $\mathbf{J}_{ij} = \frac{\partial \mathbf{e}_{ij}(\mathbf{x})}{\partial \mathbf{x}}$

# Jacobians and Sparsity

- Error $\mathbf{e}_{ij}(\mathbf{x})$ depends only on the two parameter blocks $\mathbf{x}_i$ and $\mathbf{x}_j$

$$\mathbf{e}_{ij}(\mathbf{x}) = \mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j)$$

- The Jacobian will be zero everywhere except in the columns of $\mathbf{x}_i$ and $\mathbf{x}_j$

$$\mathbf{J}_{ij} = \left( \mathbf{0} \cdots \mathbf{0} \ \underbrace{\frac{\partial \mathbf{e}(\mathbf{x}_i)}{\partial \mathbf{x}_i}}_{\mathbf{A}_{ij}} \ \mathbf{0} \cdots \mathbf{0} \ \underbrace{\frac{\partial \mathbf{e}(\mathbf{x}_j)}{\partial \mathbf{x}_j}}_{\mathbf{B}_{ij}} \ \mathbf{0} \cdots \mathbf{0} \right)$$

# Consequences of the Sparsity

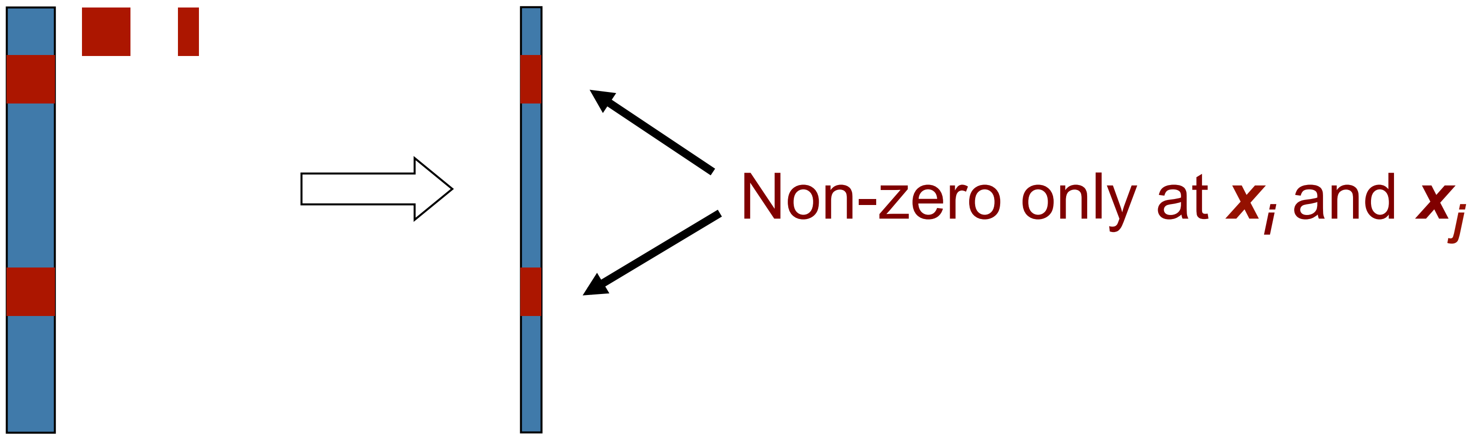
- We need to compute the coefficient vector $\mathbf{b}$ and matrix $\mathbf{H}$:

$$\begin{aligned} \mathbf{b}^T &= \sum_{ij} \mathbf{b}_{ij}^T &= \sum_{ij} \mathbf{e}_{ij}^T \boldsymbol{\Omega}_{ij} \mathbf{J}_{ij} \\ \mathbf{H} &= \sum_{ij} \mathbf{H}_{ij} &= \sum_{ij} \mathbf{J}_{ij}^T \boldsymbol{\Omega}_{ij} \mathbf{J}_{ij} \end{aligned}$$

- The sparse structure of $\mathbf{J}_{ij}$ will result in a sparse structure of $\mathbf{H}$
- This structure reflects the adjacency matrix of the graph

# Illustration of the Structure

$$\mathbf{b}_{ij} = \mathbf{J}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{e}_{ij}$$



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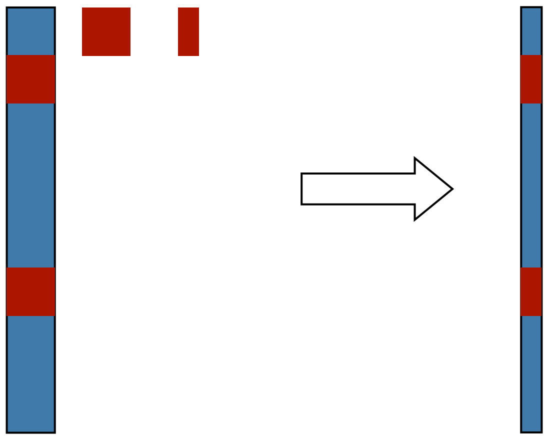
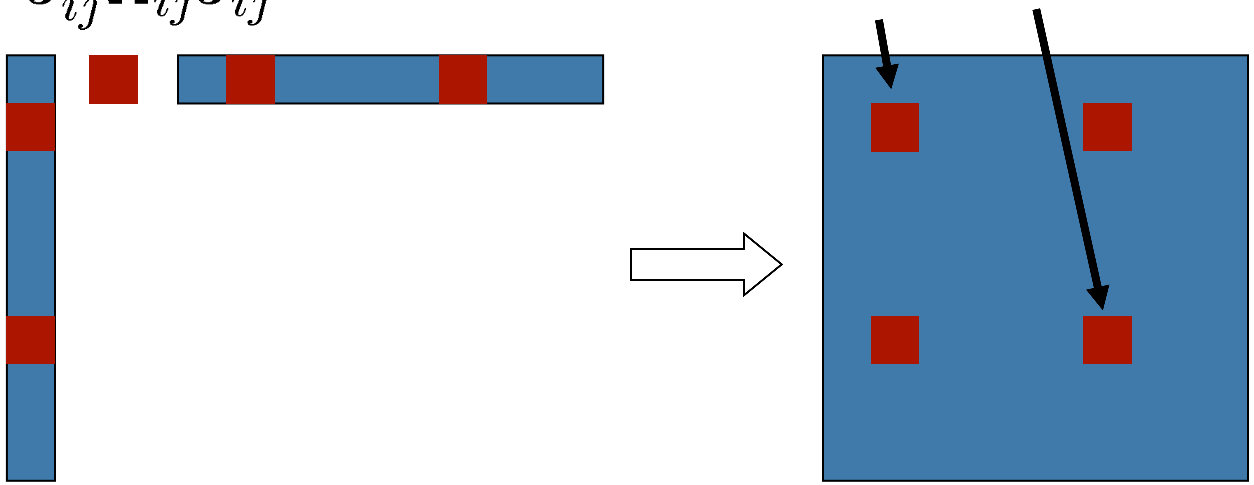


Non-zero only at $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$

$$\mathbf{H}_{ij} = \mathbf{J}_{ij}^T \boldsymbol{\Omega}_{ij} \mathbf{J}_{ij}$$

Non-zero on the main diagonal at $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$



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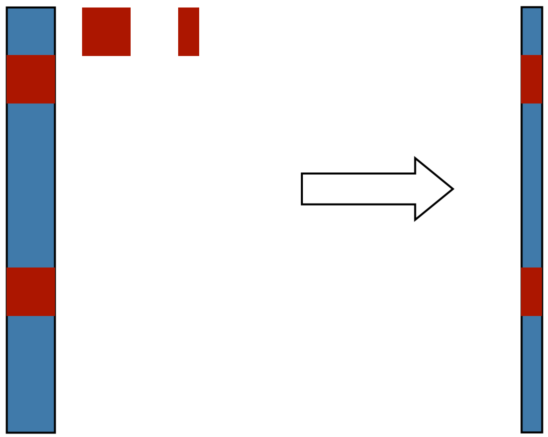

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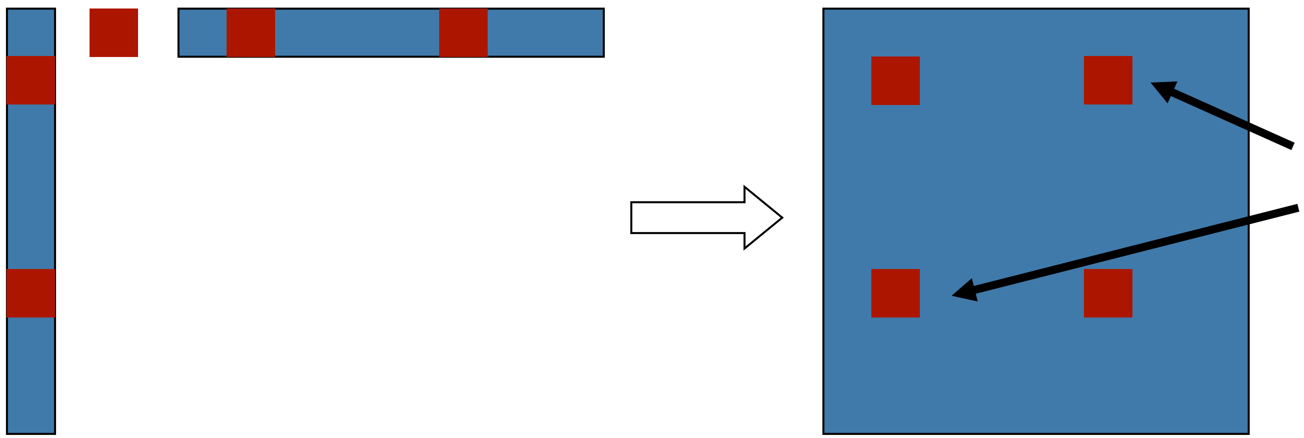
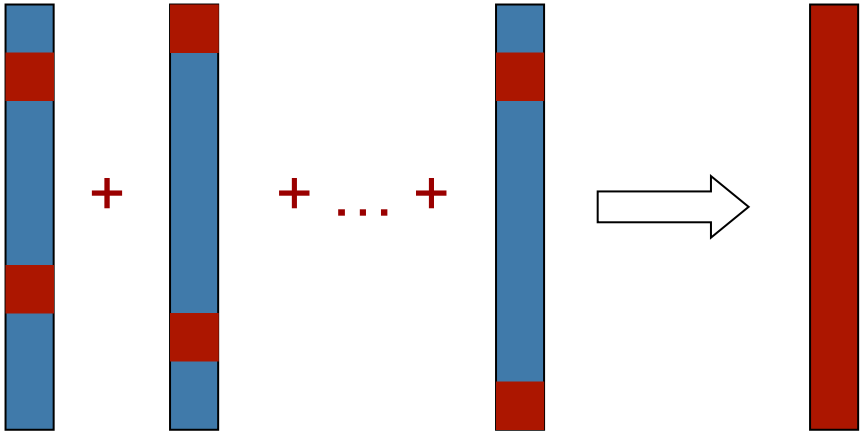


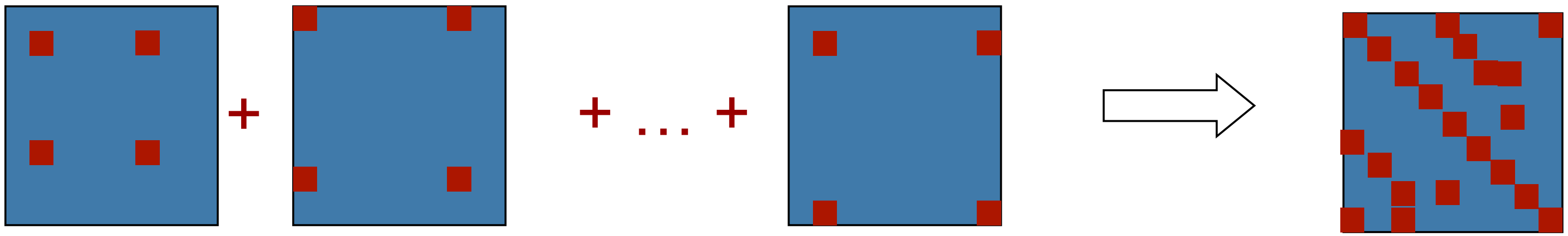
... and at the blocks *ij*,*ji*

# Illustration of the Structure

$$\mathbf{b} = \sum_{ij} \mathbf{b}_{ij}$$



$$\mathbf{H} = \sum_{ij} \mathbf{H}_{ij}$$



# The Linear System

- Vector of the states increments:

$$\Delta\mathbf{x}^T = \begin{pmatrix} \Delta\mathbf{x}_1^T & \Delta\mathbf{x}_2^T & \cdots & \Delta\mathbf{x}_n^T \end{pmatrix}$$

- Coefficient vector:

$$\mathbf{b}^T = \begin{pmatrix} \bar{\mathbf{b}}_1^T & \bar{\mathbf{b}}_2^T & \cdots & \bar{\mathbf{b}}_n^T \end{pmatrix}$$

- System matrix:

$$\mathbf{H} = \begin{pmatrix} \bar{\mathbf{H}}^{11} & \bar{\mathbf{H}}^{12} & \cdots & \bar{\mathbf{H}}^{1n} \\ \bar{\mathbf{H}}^{21} & \bar{\mathbf{H}}^{22} & \cdots & \bar{\mathbf{H}}^{2n} \\ \vdots & \ddots & & \vdots \\ \bar{\mathbf{H}}^{n1} & \bar{\mathbf{H}}^{n2} & \cdots & \bar{\mathbf{H}}^{nn} \end{pmatrix}$$

# Building the Linear System

For each constraint:

- Compute error $\mathbf{e}_{ij} = \text{t2v}(\mathbf{Z}_{ij}^{-1}(\mathbf{X}_i^{-1}\mathbf{X}_j))$
- Compute the blocks of the Jacobian:

$$\mathbf{A}_{ij} = \frac{\partial \mathbf{e}(\mathbf{x}_i, \mathbf{x}_j)}{\partial \mathbf{x}_i} \qquad \mathbf{B}_{ij} = \frac{\partial \mathbf{e}(\mathbf{x}_i, \mathbf{x}_j)}{\partial \mathbf{x}_j}$$

- Update the coefficient vector:

$$\bar{\mathbf{b}}_i^T += \mathbf{e}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{A}_{ij} \qquad \bar{\mathbf{b}}_j^T += \mathbf{e}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{B}_{ij}$$

- Update the system matrix:

$$\bar{\mathbf{H}}^{ii} += \mathbf{A}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{A}_{ij} \qquad \bar{\mathbf{H}}^{ij} += \mathbf{A}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{B}_{ij}$$

$$\bar{\mathbf{H}}^{ji} += \mathbf{B}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{A}_{ij} \qquad \bar{\mathbf{H}}^{jj} += \mathbf{B}_{ij}^T \mathbf{\Omega}_{ij} \mathbf{B}_{ij}$$

# Algorithm

**optimize($\mathbf{x}$):**

  while (!*converged*)

    $(\mathbf{H}, \mathbf{b}) = \text{buildLinearSystem}(\mathbf{x})$

    $\boldsymbol{\Delta}\mathbf{x} = \text{solveSparse}(\mathbf{H}\boldsymbol{\Delta}\mathbf{x} = -\mathbf{b})$

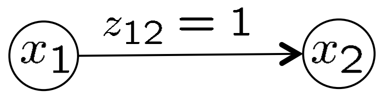
    $\mathbf{x} = \mathbf{x} + \boldsymbol{\Delta}\mathbf{x}$

  end

  *return* $\mathbf{x}$

# Example on the Blackboard

# Trivial 1D Example




- Two nodes and one observation

$$
\begin{aligned}
\mathbf{x} &= (x_1\, x_2)^T = (0\, 0) \\
\mathbf{z}_{12} &= 1 \\
\mathbf{\Omega} &= 2 \\
\mathbf{e}_{12} &= = z_{12} - (x_2 - x_1) = 1 - (0 - 0) = 1 \\
\mathbf{J}_{12} &= (1 \;\; -1) \\
\mathbf{b}_{12}^T &= \mathbf{e}_{12}^T \mathbf{\Omega}_{12} \mathbf{J}_{12} = (2 \;\; -2) \\
\mathbf{H}_{12} &= \mathbf{J}_{12}^T \mathbf{\Omega} \mathbf{J}_{12} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \\
\Delta\mathbf{x} &= -\mathbf{H}_{12}^{-1} b_{12}
\end{aligned}
$$

**BUT** $\det(\mathbf{H}) \;=\; 0$ **???**

# What Went Wrong?

- The constraint specifies a **relative constraint** between both nodes
- Any poses for the nodes would be fine as long a their relative coordinates fit
- **One node needs to be "fixed"**

$$
\begin{aligned}
\mathbf{H} &= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\Delta \mathbf{x} &= -\mathbf{H}^{-1} b_{12} \\
\Delta \mathbf{x} &= (0\,1)^T
\end{aligned}
$$

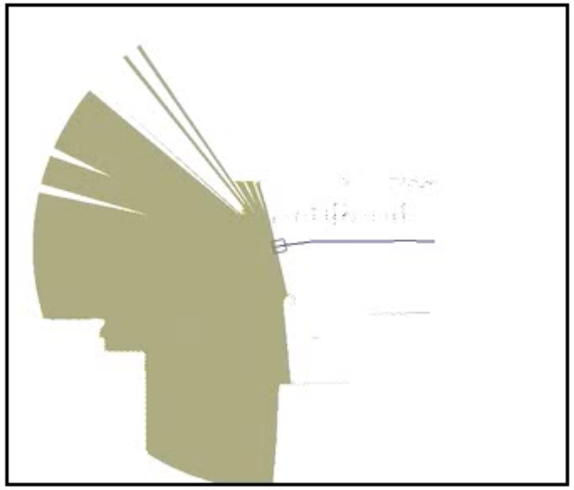
constraint that sets ***$dx_1$=0***

# Role of the Prior

- We saw that the matrix $\mathbf{H}$ has not full rank (after adding the constraints)
- The global frame had not been fixed
- Fixing the global reference frame is strongly related to the prior $p(\mathbf{x}_0)$
- A Gaussian estimate about $\mathbf{x}_0$ results in an additional constraint
- E.g., first pose in the origin:

$$\mathbf{e}(\mathbf{x}_0) = \text{t2v}(\mathbf{X}_0)$$

# Real World Examples



# Fixing a Subset of Variables

- Assume that the value of certain variables during the optimization is known a priori
- We may want to optimize all others and keep these fixed
- How?

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# Fixing a Subset of Variables

- Assume that the value of certain variables during the optimization is known a priori
- We may want to optimize all others and keep these fixed
- How?
- If a variable is not optimized, it should "disappears" from the linear system
- Construct the full system
- Suppress the rows and the columns corresponding to the variables to fix

# Why Can We Simply Suppress the Rows and Columns of the Corresponding Variables?

$$p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_\alpha \\ \boldsymbol{\mu}_\beta \end{bmatrix}, \begin{bmatrix} \Sigma_{\alpha\alpha} & \Sigma_{\alpha\beta} \\ \Sigma_{\beta\alpha} & \Sigma_{\beta\beta} \end{bmatrix}\right) = \mathcal{N}^{-1}\left(\begin{bmatrix} \boldsymbol{\eta}_\alpha \\ \boldsymbol{\eta}_\beta \end{bmatrix}, \begin{bmatrix} \Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} \\ \Lambda_{\beta\alpha} & \Lambda_{\beta\beta} \end{bmatrix}\right)$$

| | MARGINALIZATION | CONDITIONING |
|---|---|---|
| | $p(\boldsymbol{\alpha}) = \int p(\boldsymbol{\alpha}, \boldsymbol{\beta})\, d\boldsymbol{\beta}$ | $p(\boldsymbol{\alpha} \mid \boldsymbol{\beta}) = p(\boldsymbol{\alpha}, \boldsymbol{\beta}) / p(\boldsymbol{\beta})$ |
| COV. FORM | $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$<br>$\Sigma = \Sigma_{\alpha\alpha}$ | $\boldsymbol{\mu}' = \boldsymbol{\mu}_\alpha + \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)$<br>$\Sigma' = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta\alpha}$ |
| INFO. FORM | $\boldsymbol{\eta} = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\boldsymbol{\eta}_\beta$<br>$\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta}\Lambda^{-1}$ | $\boldsymbol{\eta}' = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\boldsymbol{\beta}$<br>$\Lambda' = \Lambda_{\alpha\alpha}$ |

Courtesy: R. Eustice

# Uncertainty

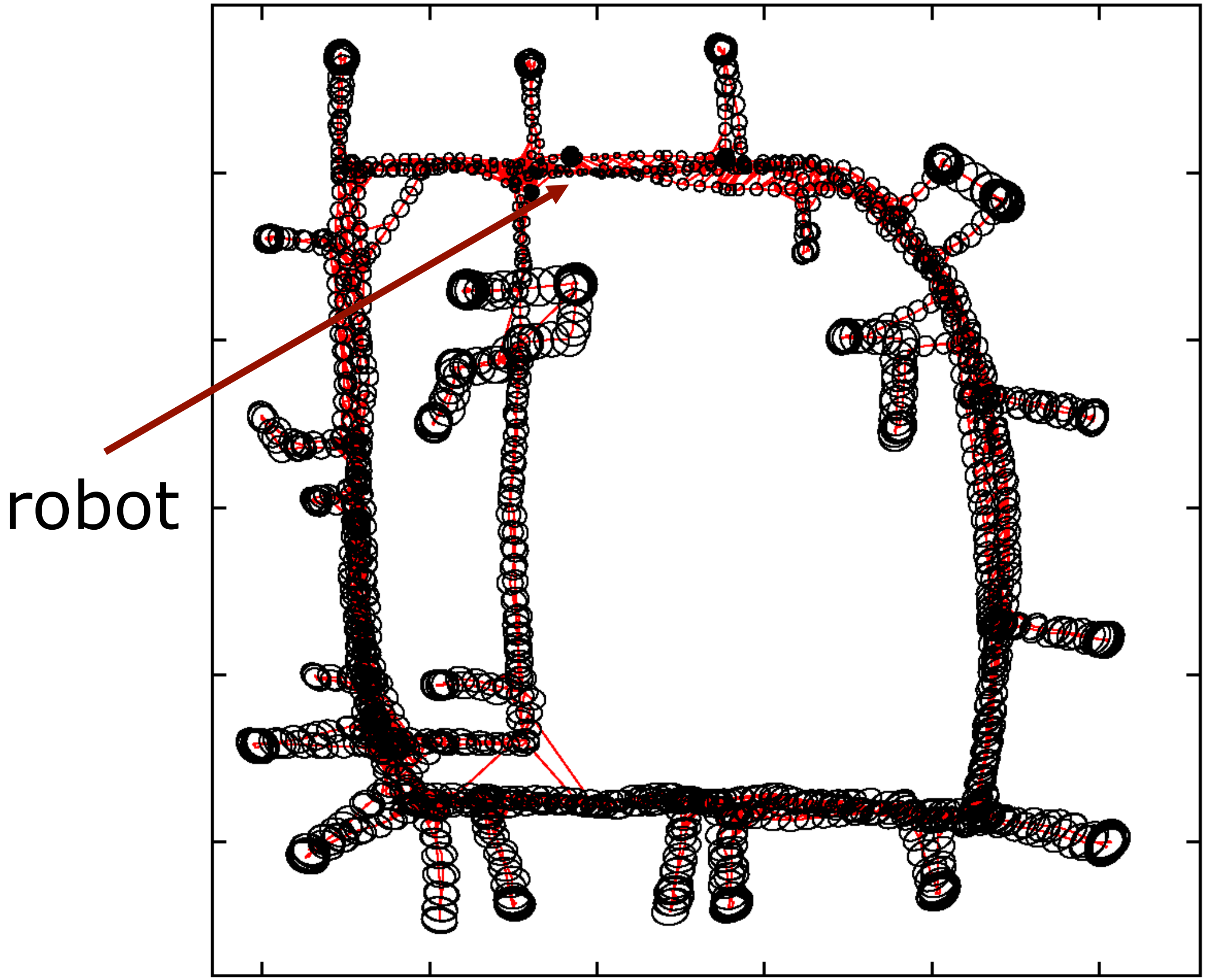
- $\mathbf{H}$ is the information matrix (given the linearization point)
- Inverting $\mathbf{H}$ results in a (dense) covariance matrix
- The diagonal blocks of the covariance matrix represent the (absolute) uncertainties of the corresponding variables

# Relative Uncertainty

To determine the relative uncertainty between two nodes $\mathbf{x}_i$ and $\mathbf{x}_j$:

- Construct the matrix $\mathbf{H}$
- Suppress the rows and the columns of $\mathbf{x}_i$ (="fixes" this variable)
- Compute the block *j,j* of the inverse
- This block will contain the covariance matrix of $\mathbf{x}_j$ w.r.t. $\mathbf{x}_i$, which has been fixed

# Example

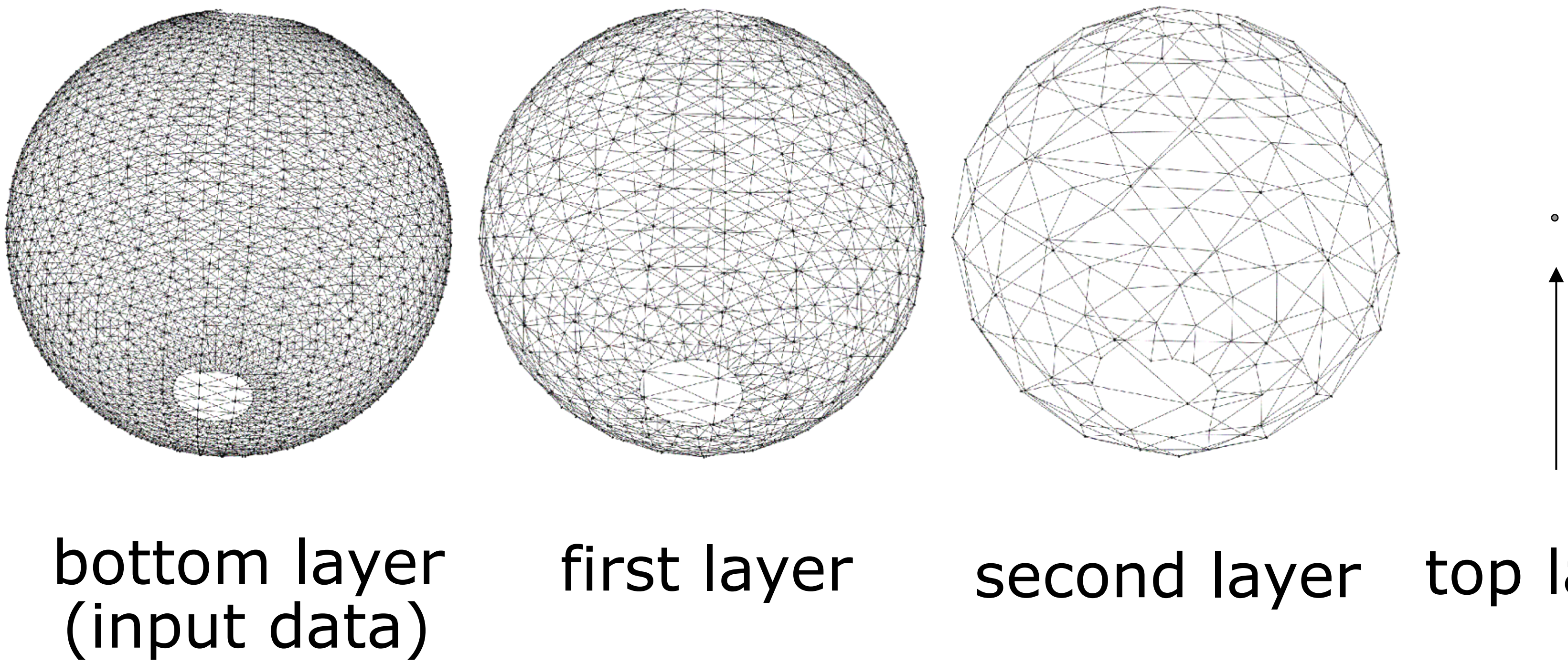


## Does all that run online?

# Does all that run online?

**… it depends on the size of the graph…**

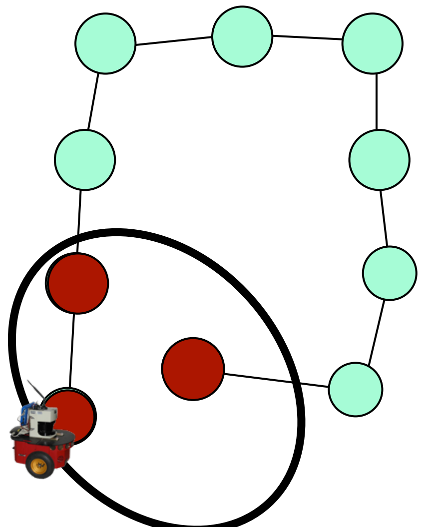
# Hierarchical Pose-Graph




“There is no need to optimize the whole graph when a new observation is obtained”

# Motivation

- Front-end seeks for loop-closures
- Requires to compare observations to all previously obtained ones
- In practice, limit search to areas in which the robot is likely to be
- This requires to know **in which parts of the graph to search for data associations**

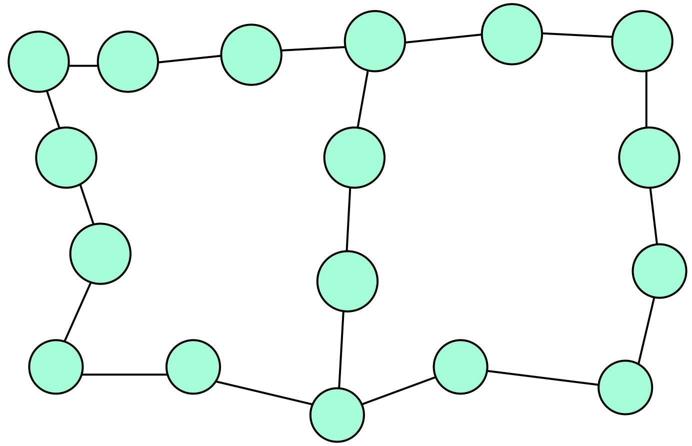


# Hierarchical Approach

- **Insight:** to find loop closures, one does not need the perfect global map
- **Idea:** correct only the core structure of the scene, not the overall graph
- The hierarchical pose-graph is a sparse approximation of the original problem
- It exploits the facts that in SLAM
  - Robot moved through the scene and it not "teleported" to locations
  - Sensors have a limited range

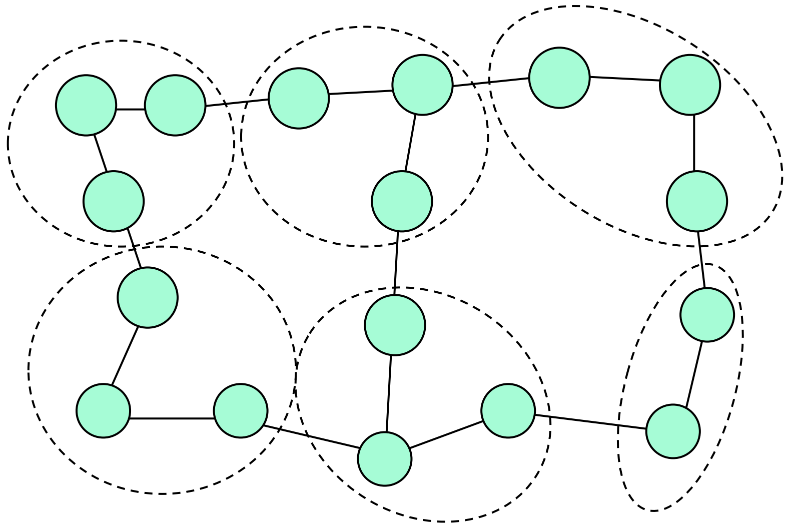
# Key Idea of the Hierarchy

- Input is the dense graph



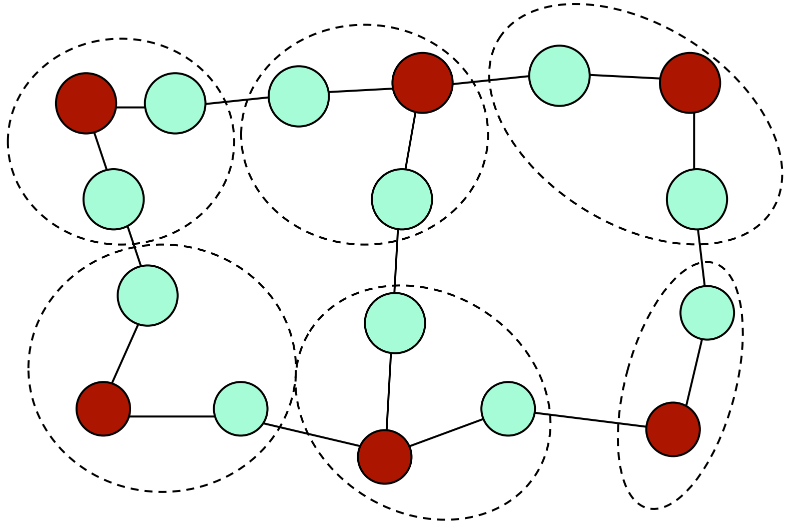
# Key Idea of the Hierarchy

- Input is the dense graph
- Group the nodes of the graph based on their local connectivity



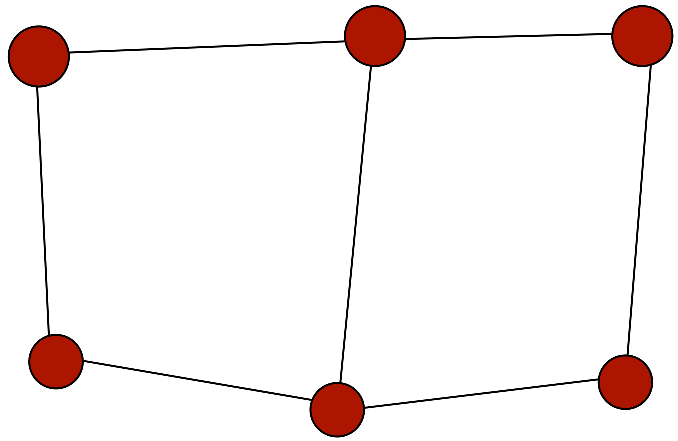
# Key Idea of the Hierarchy

- Input is the dense graph
- Group the nodes of the graph based on their local connectivity
- For each group, select one node as a "representative"



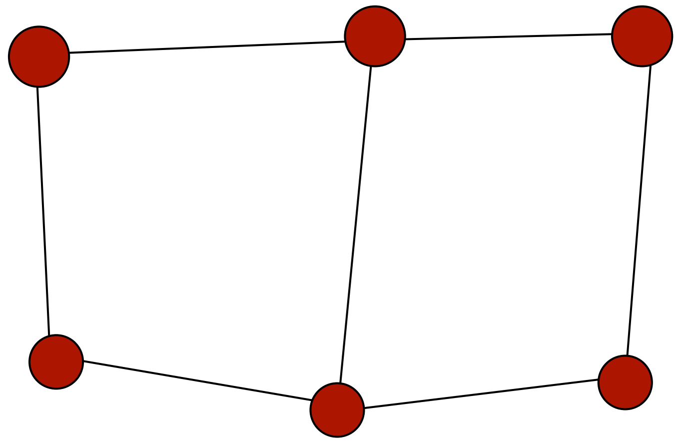
# Key Idea of the Hierarchy

- The representatives are the nodes in a new sparsified graph (upper level)



# Key Idea of the Hierarchy

- The representatives are the nodes in a new sparsified graph (upper level)
- Edges of the sparse graph are determined by the connectivity of the groups of nodes
- The parameters of the sparse edges are estimated via local optimization



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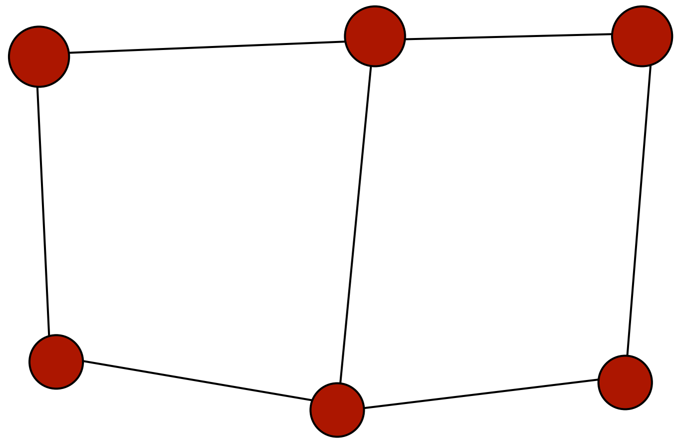
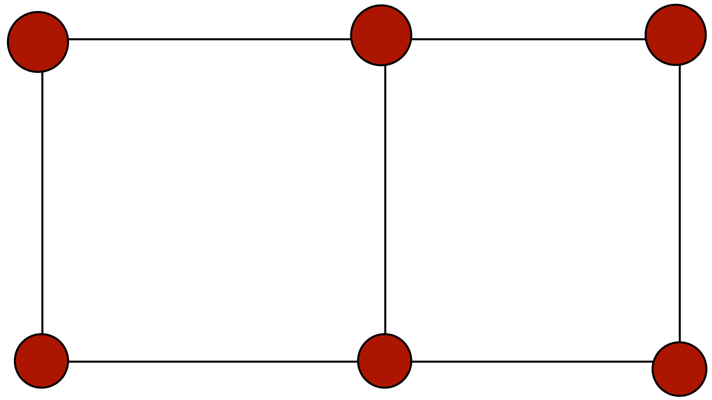


Process is repeated recursively

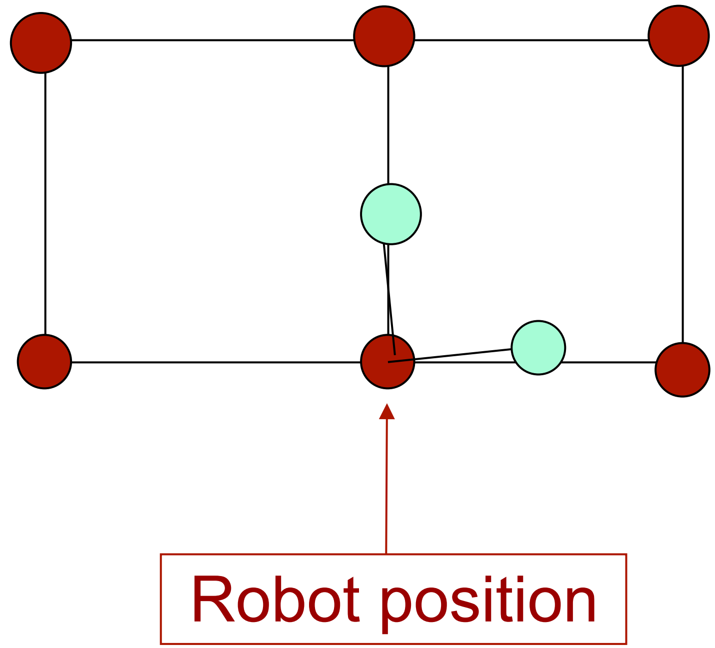
# Key Idea of the Hierarchy

- Only the upper level of the hierarchy is optimized completely



# Key Idea of the Hierarchy

- Only the upper level of the hierarchy is optimized completely
- The changes are propagated to the bottom levels only close to the current robot position
- Only this part of the graph is relevant for finding constraints

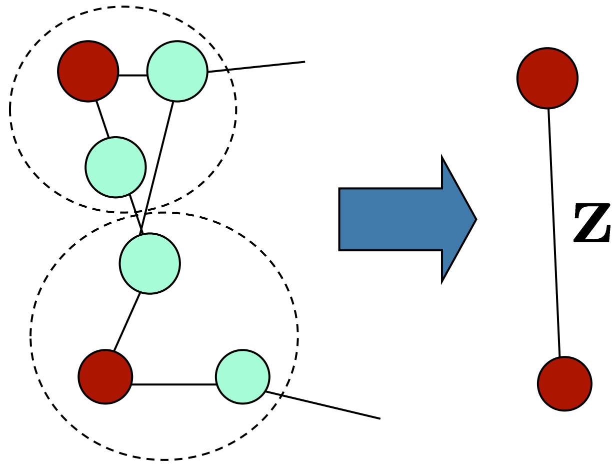


# Construction of the Hierarchy

- When and how to generate a new group?
  - A (simple) distance-based decision
  - The first node of a new group is the representative
- When to propagate information downwards?
  - Only when there are inconsistencies
- How to construct an edge in the sparsified graph?
  - Next slides
- How to propagate information downwards?
  - Next slides

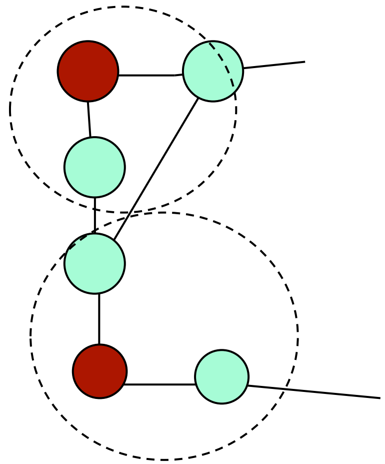
# Determining Edge Parameters

- Given two connected groups
- How to compute a virtual observation $\mathbf{z}$ and the information matrix $\Omega$ for the new edge?



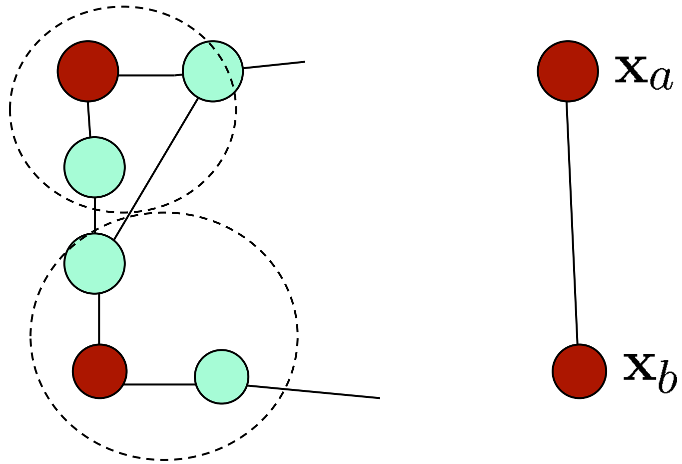
# Determining Edge Parameters

- Optimize the two sub-groups jointly but independently from the rest



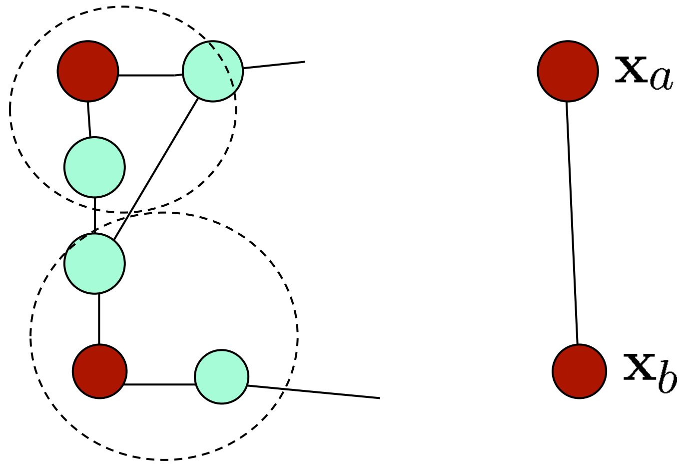
# Determining Edge Parameters

- Optimize the two sub-groups jointly but independently from the rest
- The observation is the relative transformation between the two representatives



# Determining Edge Parameters

- Optimize the two sub-groups jointly but independently from the rest
- The observation is the relative transformation between the two representatives
- The information matrix is computed from the diagonal block of the matrix ***H***

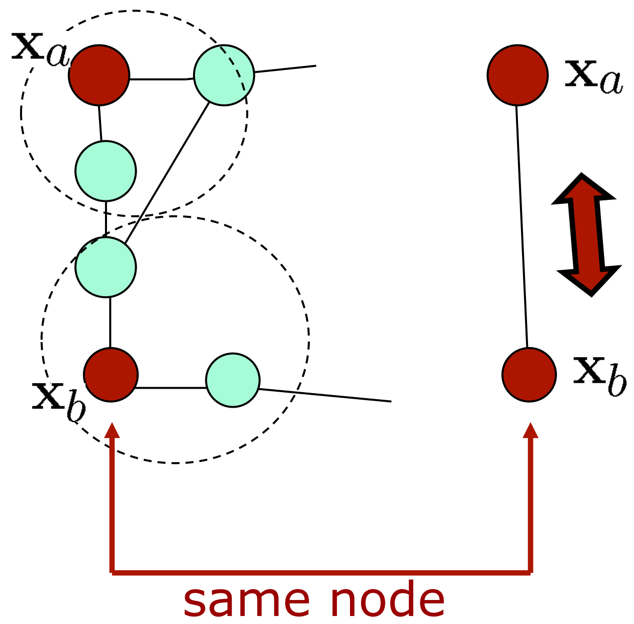



Inverse of the *[b,b]* block of ***H**$^{-1}$*

$$\Omega_{ab} = (\mathbf{H}^{-1}_{[b,b]})^{-1}$$

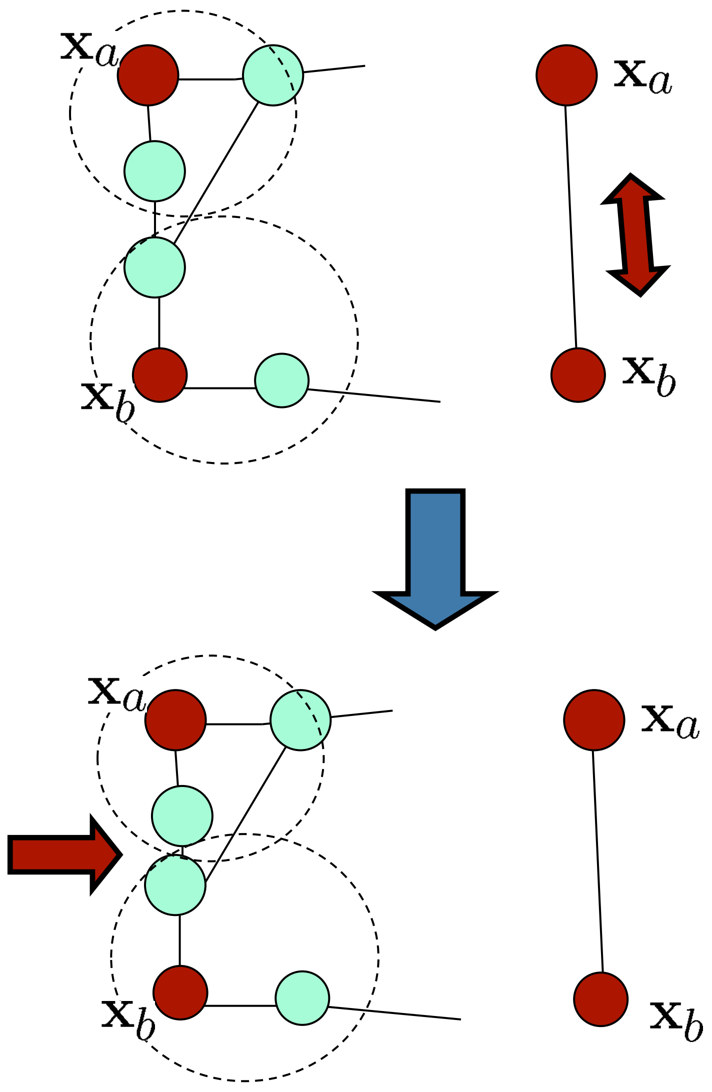
# Propagating Information Downwards

- All representatives are nodes from the lower (bottom) level



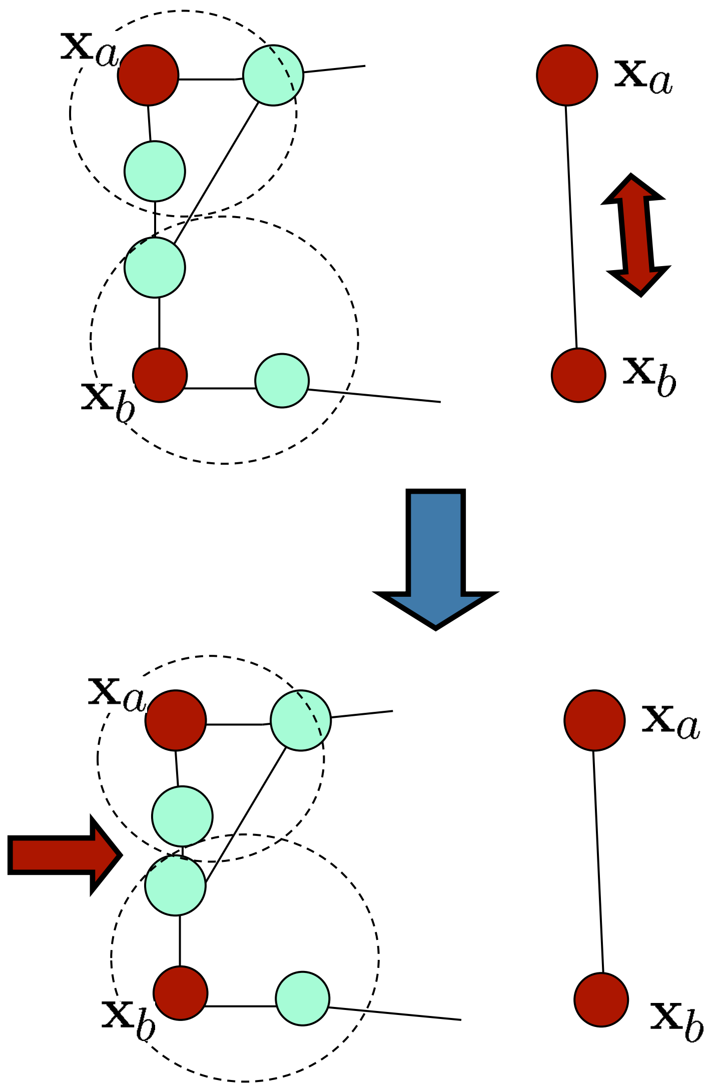
# Propagating Information Downwards

- All representatives are nodes from the lower (bottom) level
- Information is propagated downwards by transforming the group at the lower level using a rigid body transformation



# Propagating Information Downwards

- All representatives are nodes from the lower (bottom) level
- Information is propagated downwards by transforming the group at the lower level using a rigid body transformation
- Only if the lower level becomes inconsistent, optimize at the lower level



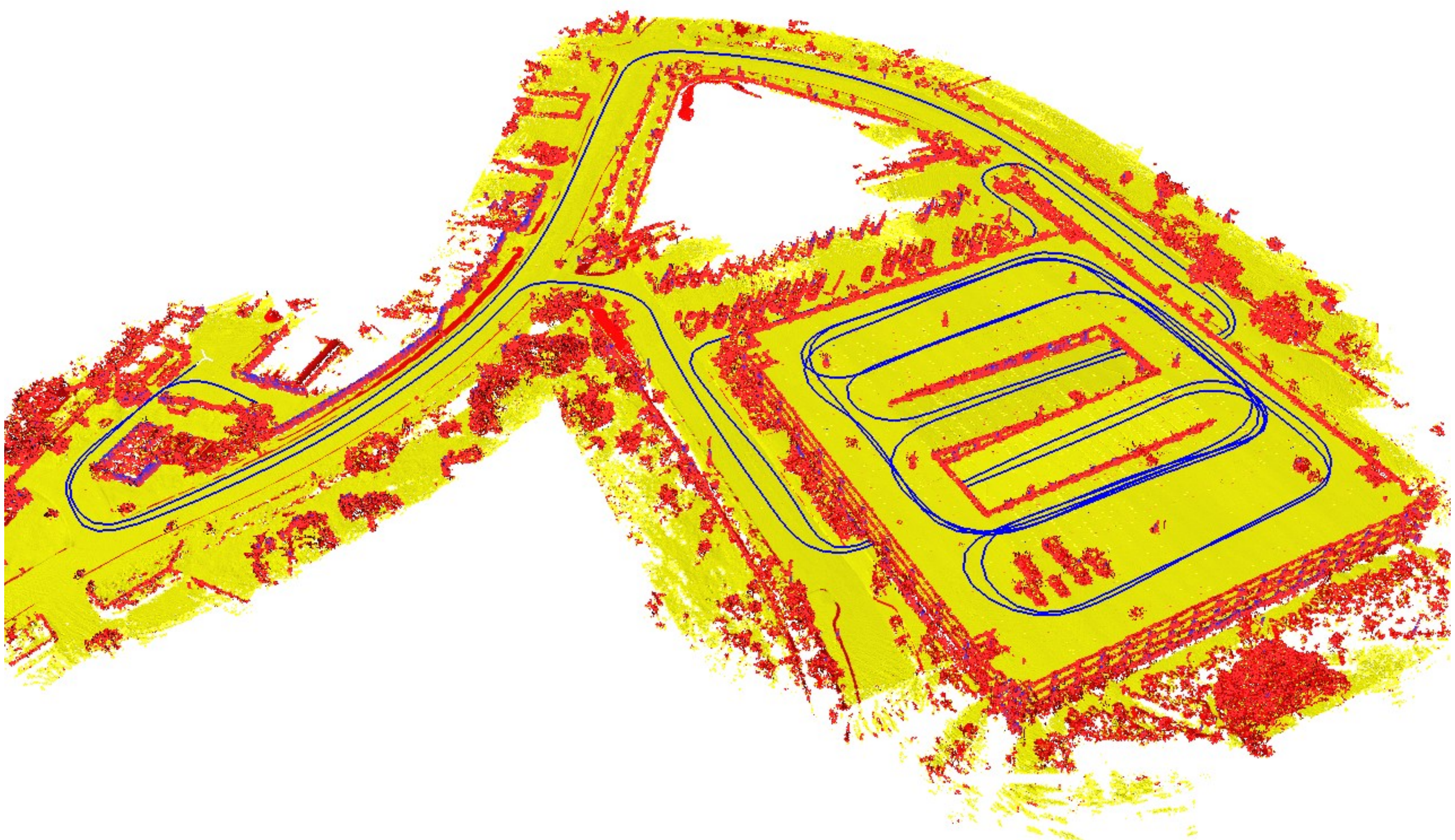
# For the Best Possible Map…

- Run the optimization on the lowest level (at the end)
- For offline processing with all constraints, the hierarchy helps convergence faster in case of large errors
- In this case, one pass up the tree (to construct the edges) followed by one pass down the tree is sufficient

# Stanford Garage

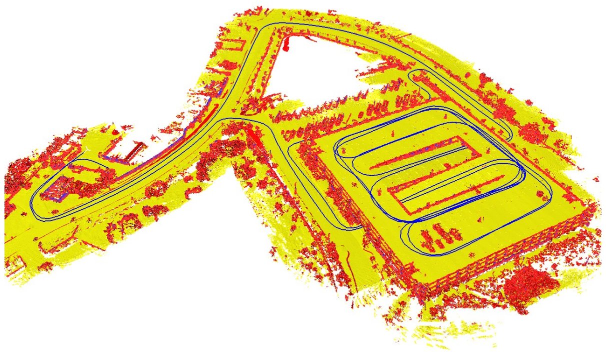
- Parking garage at Stanford University
- Nested loops, trajectory of ~7,000m

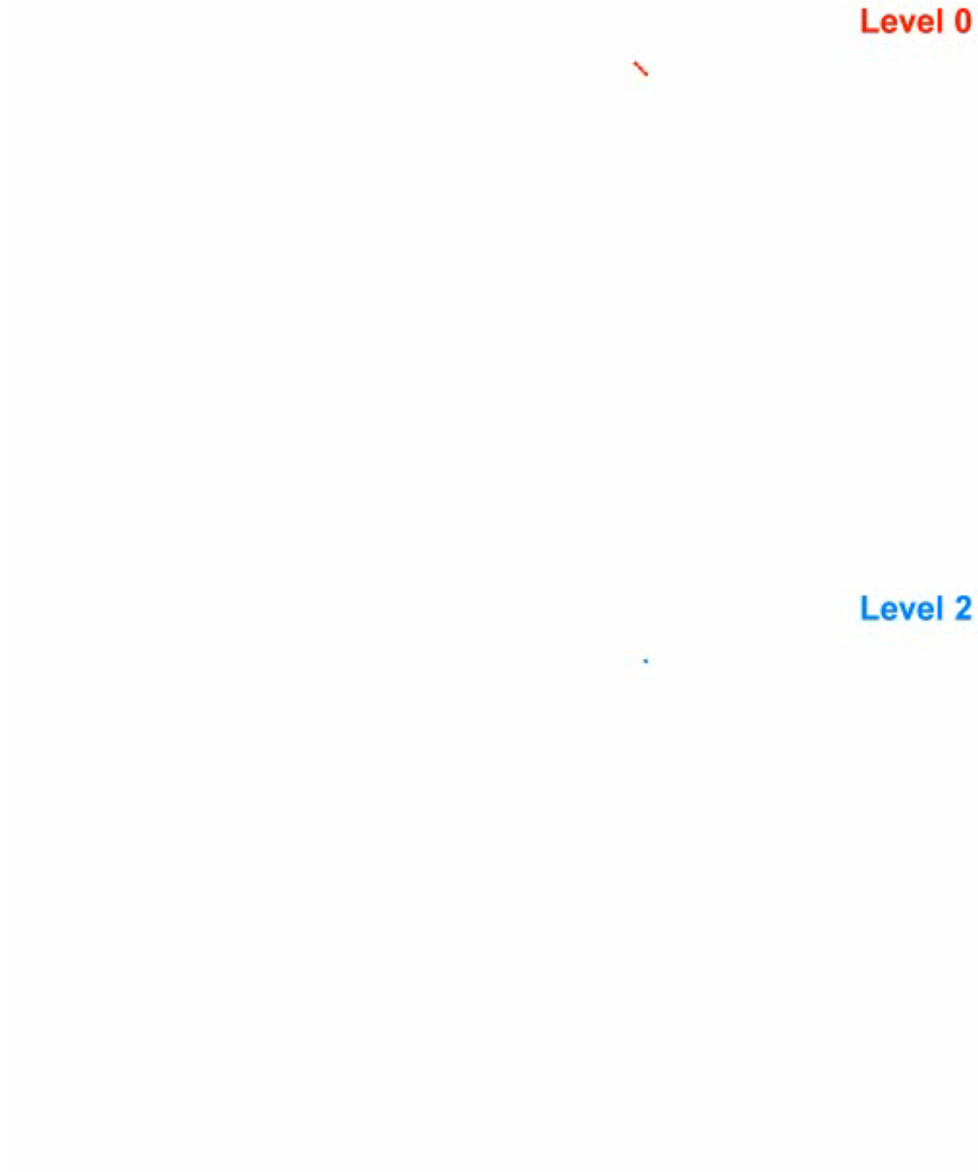
# Stanford Garage Result



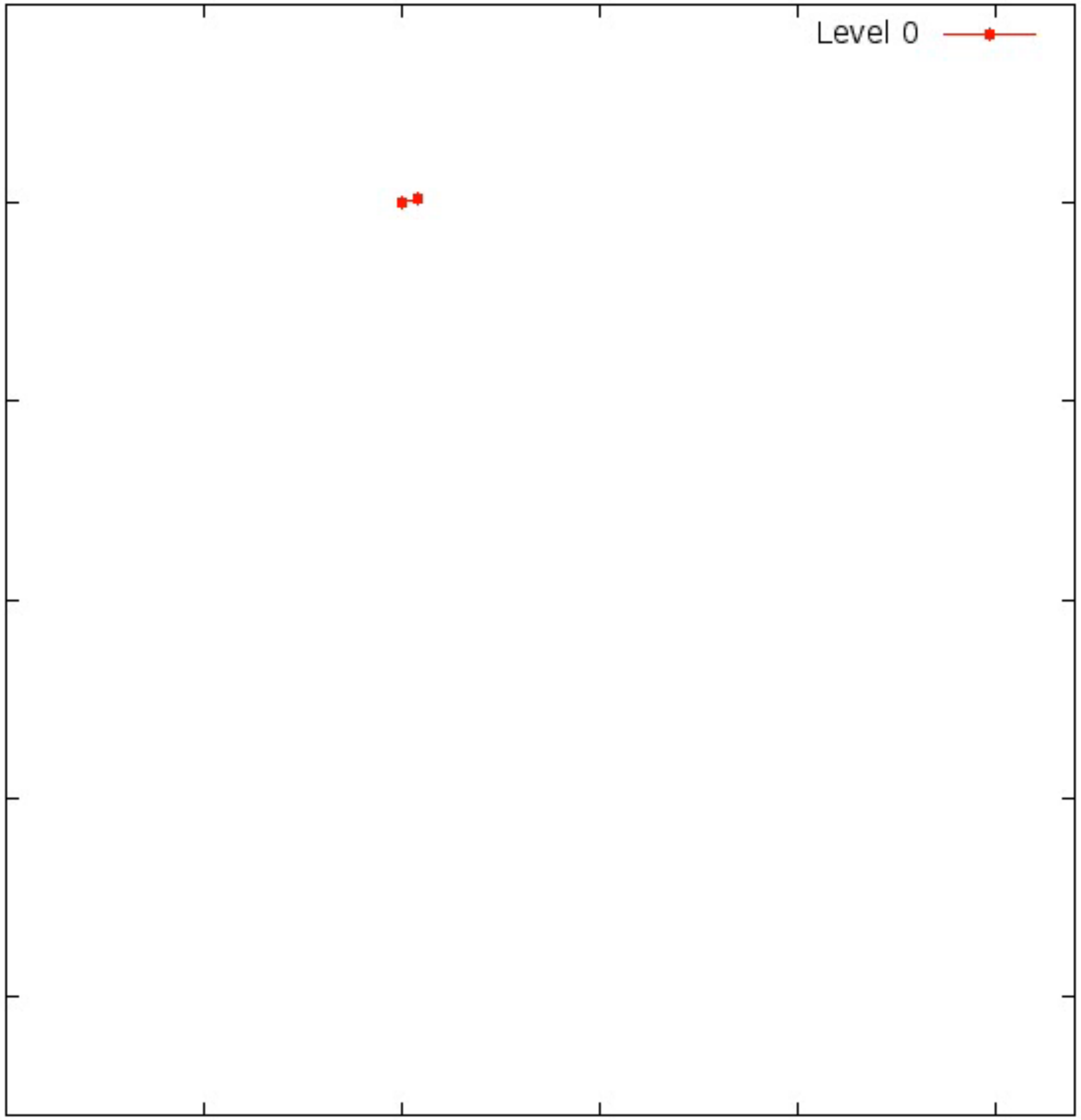
- Parking garage at Stanford University
- Nested loops, trajectory of ~7,000m

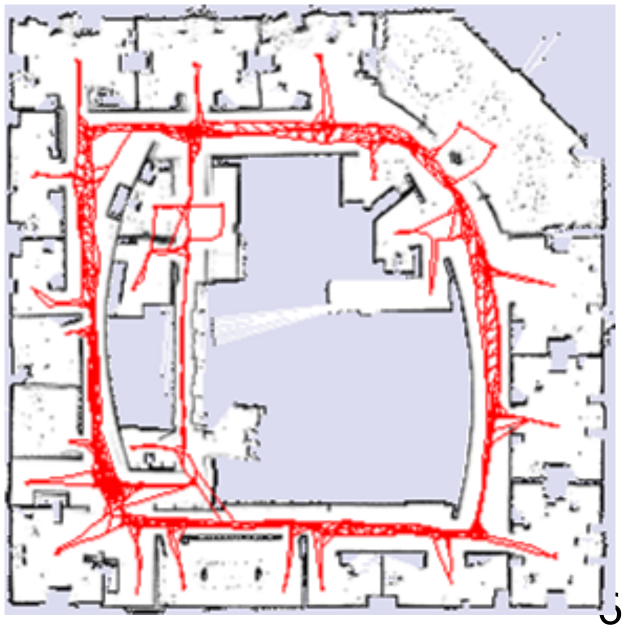
# Stanford Garage Video





# Intel Research Lab Video



# Consistency

- How well does the top level in the hierarchy represent the original input?

# Consistency

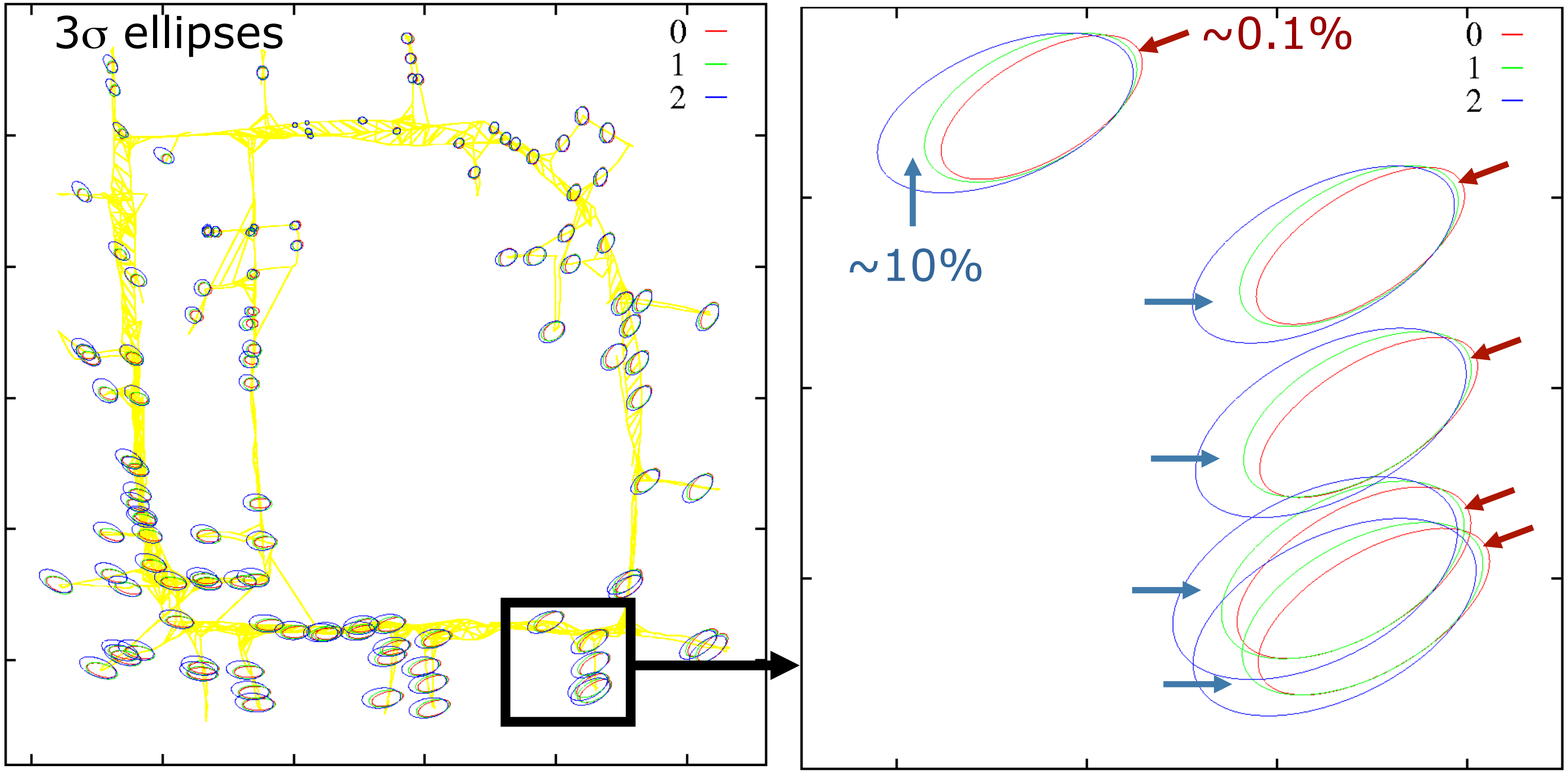
- How well does the top level in the hierarchy represent the original input?
- Probability mass of the marginal distribution in the highest level vs. the one of the true estimate (original problem, lowest level)

| | Prob. mass not covered | Prob. mass outside |
|---|---|---|
| Intel | 0.10% | 10.18% |
| W-10000 | 2.53% | 24.05% |
| Stanford | 0.01% | 7.88% |
| Sphere | 2.75% | 10.21% |

low risk of becoming overly confident

one does not ignore too much information

# Consistency




- Red: overly confident (~0.1% prob. mass)
- Blue: under confident (~10% prob. mass)

# Conclusions

- The back-end part of the SLAM problem can be effectively solved with Gauss-Newton
- The $\mathbf{H}$ matrix is typically sparse
- This sparsity allows for efficiently solving the linear system
- One of the state-of-the-art solutions for computing maps
- Hierarchical pose-graph for computing approximate solutions online

# Literature

## Least Squares SLAM

- Grisetti, Kümmerle, Stachniss, Burgard: "A Tutorial on Graph-based SLAM", 2010

## Hierarchical Approach

- Grisetti, Kümmerle, Stachniss, Frese, and Hertzberg: "Hierarchical Optimization on Manifolds for Online 2D and 3D Mapping"
- Code: http://openslam.org/hog-man.html