# Robot Mapping

## Extended Information Filter

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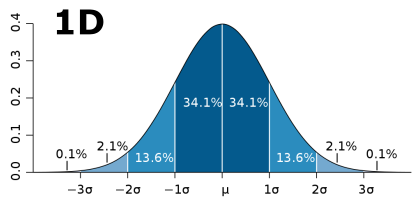
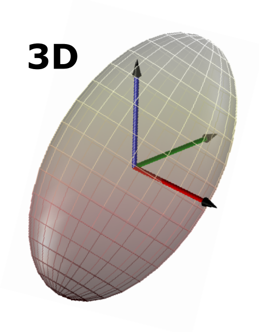
## Gaussians

- Gaussian described by **moments** $\mu, \Sigma$

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$






## Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** $\Omega$ and **information vector** $\xi$



## Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** $\Omega$

$$\Omega = \Sigma^{-1}$$

- and **information vector** $\xi$

$$\xi = \Sigma^{-1}\mu$$

## Complete Parameterizations

moments

$$\Sigma = \Omega^{-1}$$
$$\mu = \Omega^{-1}\xi$$

canonical

$$\Omega = \Sigma^{-1}$$
$$\xi = \Sigma^{-1}\mu$$



## Towards the Information Form

$$\begin{aligned} p(x) & \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \end{aligned}$$



## Towards the Information Form

$$\begin{aligned} p(x) & \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(\underline{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}\Big) \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \end{aligned}$$



## Towards the Information Form

$$\begin{aligned} p(x) & \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(\underline{-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu} - \underline{\frac{1}{2}\mu^T\Sigma^{-1}\mu}\Big) \\ &= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\ &\qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \end{aligned}$$

## Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
&= \underline{\det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\Big)} \\
&\qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \\
&= \eta \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big)
\end{aligned}
$$



## Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
&\qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \\
&= \eta \exp\Big(-\frac{1}{2}x^T\underline{\Sigma^{-1}}x + x^T\underline{\Sigma^{-1}}\mu\Big) \\
&= \eta \exp\Big(-\frac{1}{2}x^T\Omega x + x^T\xi\Big)
\end{aligned}
$$



## Dual Representation

$$
p(x) = \frac{\exp(-\frac{1}{2}\mu^T\xi)}{\det(2\pi\Omega^{-1})^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}x^T\Omega x + x^T\xi\Big)
$$

canonical parameterization

$$
p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)
$$

moments parameterization



## Marginalization and Conditioning

$$
p(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathcal{N}\Big(\begin{bmatrix}\boldsymbol{\mu}_\alpha \\ \boldsymbol{\mu}_\beta\end{bmatrix}, \begin{bmatrix}\Sigma_{\alpha\alpha} & \Sigma_{\alpha\beta} \\ \Sigma_{\beta\alpha} & \Sigma_{\beta\beta}\end{bmatrix}\Big) = \mathcal{N}^{-1}\Big(\begin{bmatrix}\boldsymbol{\eta}_\alpha \\ \boldsymbol{\eta}_\beta\end{bmatrix}, \begin{bmatrix}\Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} \\ \Lambda_{\beta\alpha} & \Lambda_{\beta\beta}\end{bmatrix}\Big)
$$

| | MARGINALIZATION $p(\boldsymbol{\alpha}) = \int p(\boldsymbol{\alpha},\boldsymbol{\beta})\, d\boldsymbol{\beta}$ | CONDITIONING $p(\boldsymbol{\alpha} \mid \boldsymbol{\beta}) = p(\boldsymbol{\alpha},\boldsymbol{\beta})/p(\boldsymbol{\beta})$ |
|---|---|---|
| COV. FORM | $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$<br>$\Sigma = \Sigma_{\alpha\alpha}$ | $\boldsymbol{\mu}' = \boldsymbol{\mu}_\alpha + \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)$<br>$\Sigma' = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta\alpha}$ |
| INFO. FORM | $\boldsymbol{\eta} = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\boldsymbol{\eta}_\beta$<br>$\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}$ | $\boldsymbol{\eta}' = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\boldsymbol{\beta}$<br>$\Lambda' = \Lambda_{\alpha\alpha}$ |

trivial

expensive

Courtesy: R. Eustice

## From the Kalman Filter to the Information Filter

- Two parameterization for Gaussian
- Same expressiveness
- Marginalization and conditioning have different complexities
- We learned about Gaussian filtering with the Kalman filter in Chapter 4
- Kalman filtering in information from is called information filtering



## Kalman Filter Algorithm

**Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$\bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t$
$\bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + R_t$

$K_t = \bar{\Sigma}_t\ C_t^T (C_t\ \bar{\Sigma}_t\ C_t^T + Q_t)^{-1}$
$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\ \bar{\mu}_t)$
$\Sigma_t = (I - K_t\ C_t)\ \bar{\Sigma}_t$
*return* $\mu_t, \Sigma_t$



## Prediction Step (1)

- Transform $\bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + R_t$
- Using $\Sigma_{t-1} = \Omega_{t-1}^{-1}$
- Leads to

$$\bar{\Omega}_t = (A_t\ \Omega_{t-1}^{-1}\ A_t^T + R_t)^{-1}$$



## Prediction Step (2)

- Transform $\bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t$
- Using $\bar{\mu}_{t-1} = \Omega_{t-1}^{-1} \xi_{t-1}$
- Leads to

$$\begin{aligned} \bar{\xi}_t &= \bar{\Omega}_t (A_t\ \mu_{t-1} + B_t\ u_t) \\ &= \bar{\Omega}_t (A_t\ \Omega_{t-1}^{-1} \xi_{t-1} + B_t\ u_t) \end{aligned}$$

## Information Filter Algorithm

$$\textbf{Information\_filter}(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)\textbf{:}$$

$$\bar{\Omega}_t = (A_t\ \Omega_{t-1}^{-1}\ A_t^T + R_t)^{-1}$$
$$\bar{\xi}_t = \bar{\Omega}_t (A_t\ \Omega_{t-1}^{-1}\ \xi_{t-1} + B_t\ u_t)$$



## Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
bel(x_t) &= \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t) \\
&= \eta'\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\ Q_t^{-1}\ (z_t - C_t x_t)\right)\ \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\ \bar{\Sigma}_t^{-1}\ (x_t - \bar{\mu}_t)\right)
\end{aligned}
$$



## Correction Step

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bel(x_t) &= \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t) \\
&= \eta'\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\ Q_t^{-1}\ (z_t - C_t x_t)\right)\ \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\ \bar{\Sigma}_t^{-1}\ (x_t - \bar{\mu}_t)\right) \\
&= \eta'\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\ Q_t^{-1}\ (z_t - C_t x_t) - \frac{1}{2}\,(x_t - \bar{\mu}_t)^T\ \bar{\Sigma}_t^{-1}\ (x_t - \bar{\mu}_t)\right)
\end{aligned}
$$



## Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
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&= \eta'\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\ Q_t^{-1}\ (z_t - C_t x_t) - \frac{1}{2}\,(x_t - \bar{\mu}_t)^T\ \bar{\Sigma}_t^{-1}\ (x_t - \bar{\mu}_t)\right) \\
&= \eta''\ \exp\left(-\frac{1}{2}\,x_t^T\ C_t^T\ Q_t^{-1}\ C_t\ x_t + x_t^T\ C_t^T\ Q_t^{-1}\ z_t - \frac{1}{2}\,x_t^T\ \bar{\Omega}_t x_t + x_t^T \bar{\xi}_t\right)
\end{aligned}
$$

## Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
bel(x_t) &= \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t) \\
&= \eta' \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t)\right)\, \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right) \\
&= \eta' \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t) - \frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right) \\
&= \eta'' \exp\left(-\frac{1}{2}\,x_t^T\, C_t^T\, Q_t^{-1}\, C_t\, x_t + x_t^T\, C_t^T\, Q_t^{-1}\, z_t - \frac{1}{2}\,x_t^T\, \bar{\Omega}_t x_t + x_t^T \bar{\xi}_t\right) \\
&= \eta'' \exp\left(-\frac{1}{2}\,x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t]}_{\Omega_t}\, x_t + x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t]}_{\xi_t}\right)
\end{aligned}
$$



## Correction Step

- This results in a simple update rule

$$
bel(x_t) = \eta \exp\left(-\frac{1}{2}\,x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t]}_{\Omega_t}\, x_t + x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t]}_{\xi_t}\right)
$$

$$
\begin{aligned}
\Omega_t &= C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t \\
\xi_t &= C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t
\end{aligned}
$$



## Information Filter Algorithm

1: **Information_filter**$(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)$**:**

2: $\bar{\Omega}_t = (A_t\, \Omega_{t-1}^{-1}\, A_t^T + R_t)^{-1}$
3: $\bar{\xi}_t = \bar{\Omega}_t (A_t\, \Omega_{t-1}^{-1}\, \xi_{t-1} + B_t\, u_t)$

4: $\Omega_t = C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t$
5: $\xi_t = C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t$
6: *return* $\xi_t, \Omega_t$



## Prediction and Correction

- Prediction

$$
\begin{aligned}
\bar{\Omega}_t &= (A_t\, \Omega_{t-1}^{-1}\, A_t^T + R_t)^{-1} \\
\bar{\xi}_t &= \bar{\Omega}_t (A_t\, \Omega_{t-1}^{-1}\, \xi_{t-1} + B_t\, u_t)
\end{aligned}
$$

- Correction

$$
\begin{aligned}
\Omega_t &= C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t \\
\xi_t &= C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t
\end{aligned}
$$

**Discuss differences to the KF!**

## Complexity

- Kalman filter
  - Efficient prediction step: $\mathcal{O}(n^2)^*$
  - Costly correction step: $\mathcal{O}(n^2 + k^{2.4})$
- Information filter
  - Costly prediction step: $\mathcal{O}(n^{2.4})$
  - Efficient correction step: $\mathcal{O}(n^2)^*$
- Transformation between both parameterizations is costly: $\mathcal{O}(n^{2.4})$

*Potentially faster, especially for SLAM; depending on type of controls and observations



## Extended Information Filter

- As the Kalman filter, the information filter suffers from the linear models
- The extended information filter (EIF) uses a similar trick as the EKF
- Linearization of the motion and observation function



## Linearization of the EIF

- Taylor approximation analog to the EKF (see Chapter 3)

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + G_t\,(x_{t-1} - \mu_{t-1}) \\ h(x_t) &\approx h(\bar{\mu}_t) + H_t\,(x_t - \bar{\mu}_t) \end{aligned}$$

- with the Jacobians $G_t$ and $H_t$



## Prediction: From EKF of EIF

- Substitution of the moments brings us from the EKF

$$\begin{aligned} \bar{\Sigma}_t &= G_t\,\Sigma_{t-1}\,G_t^T + R_t \\ \bar{\mu}_t &= g(u_t, \mu_{t-1}) \end{aligned}$$

- to the EIF

$$\begin{aligned} \bar{\Omega}_t &= (G_t\,\Omega_{t-1}^{-1}\,G_t^T + R_t)^{-1} \\ \bar{\xi}_t &= \bar{\Omega}_t\,g(u_t, \Omega_{t-1}^{-1}\,\xi_{t-1}) \end{aligned}$$

## Prediction: From EKF of EIF

**Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$$

**Extended_information_filter**$(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)$**:**

$$\mu_{t-1} = \Omega_{t-1}^{-1}\, \xi_{t-1}$$
$$\bar{\Omega}_t = (G_t\, \Omega_{t-1}^{-1}\, G_t^T + R_t)^{-1}$$
$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\xi}_t = \bar{\Omega}_t\, \bar{\mu}_t$$



## Correction Step of the EIF

- As from the KF to IF transition, use substitute the moments in the measurement update

$$bel(x_t) = \eta \exp\left(-\frac{1}{2}\,(z_t - h(\bar{\mu}_t) - H_t\,(x_t - \bar{\mu}_t))^T\, Q_t^{-1}\, (z_t - h(\bar{\mu}_t) - H_t\,(x_t - \bar{\mu}_t)) - \frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (x_t - \bar{\mu}_t)\right)$$

- This leads to

$$\Omega_t = \bar{\Omega}_t + H_t^T\, Q_t^{-1}\, H_t$$
$$\xi_t = \bar{\xi}_t + H_t^T\, Q_t^{-1}\, (z_t - h(\bar{\mu}_t) + H_t\, \bar{\mu}_t)$$



## Extended Information Filter

**Extended_information_filter**$(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)$**:**

$$\mu_{t-1} = \Omega_{t-1}^{-1}\, \xi_{t-1}$$
$$\bar{\Omega}_t = (G_t\, \Omega_{t-1}^{-1}\, G_t^T + R_t)^{-1}$$
$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\xi}_t = \bar{\Omega}_t\, \bar{\mu}_t$$
$$\Omega_t = \bar{\Omega}_t + H_t^T\, Q_t^{-1}\, H_t$$
$$\xi_t = \bar{\xi}_t + H_t^T\, Q_t^{-1}\, (z_t - h(\bar{\mu}_t) + H_t\, \bar{\mu}_t)$$
$$\textit{return}\ \xi_t, \Omega_t$$



## EIF vs. EKF

- The EIF is the EKF in information form
- Complexities of the prediction and correction steps differ
- Same expressiveness than the EKF
- Unscented transform can also be used
- Reported to be numerically more stable than the EKF
- In practice, the EKF is more popular than the EIF

## Summary

- Gaussians can also be represented using the canonical parameterization
- Allow for filtering in information form
- Information filter vs. Kalman filter
- KF: efficient prediction, slow correction
- IF: slow prediction, efficient correction
- The application determines which filter is the better choice!



## Literature

**Extended Information Filter**

- Thrun et al.: "Probabilistic Robotics", Chapter 3.5