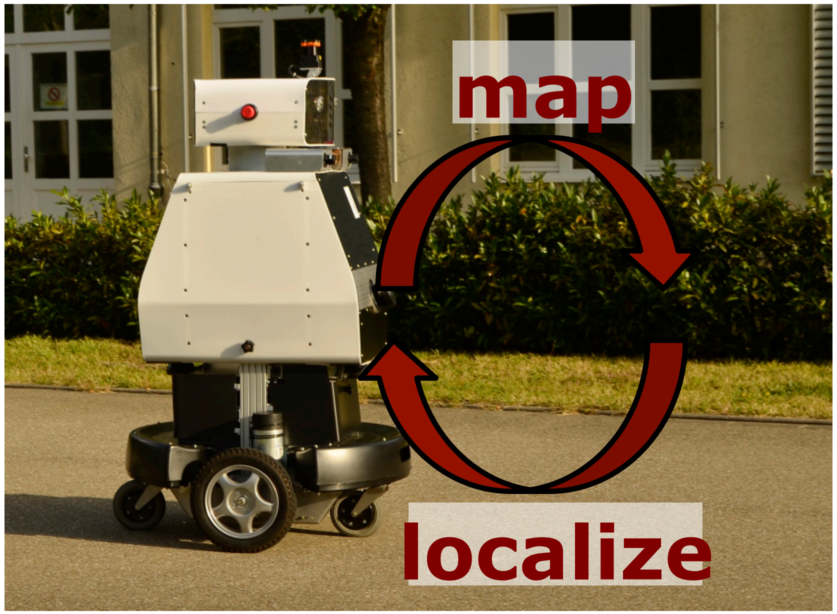
Robot Mapping

EKF SLAM

Cyrill Stachniss

Simultaneous Localization and Mapping (SLAM)

- Building a map and locating the robot in the map at the same time
- Chicken-or-egg problem



Definition of the SLAM Problem

Given

- The robot’s controls

$$u_{1:T} = \{u_1, u_2, u_3, \ldots, u_T\}$$

- Observations

$$z_{1:T} = \{z_1, z_2, z_3, \ldots, z_T\}$$

Wanted

- Map of the environment

$$m$$

- Path of the robot

$$x_{0:T} = \{x_0, x_1, x_2, \ldots, x_T\}$$

Three Main Paradigms

Kalman filter	Particle filter	Graph-based

Bayes Filter

- Recursive filter with prediction and correction step

- **Prediction**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\, dx_{t-1}$$

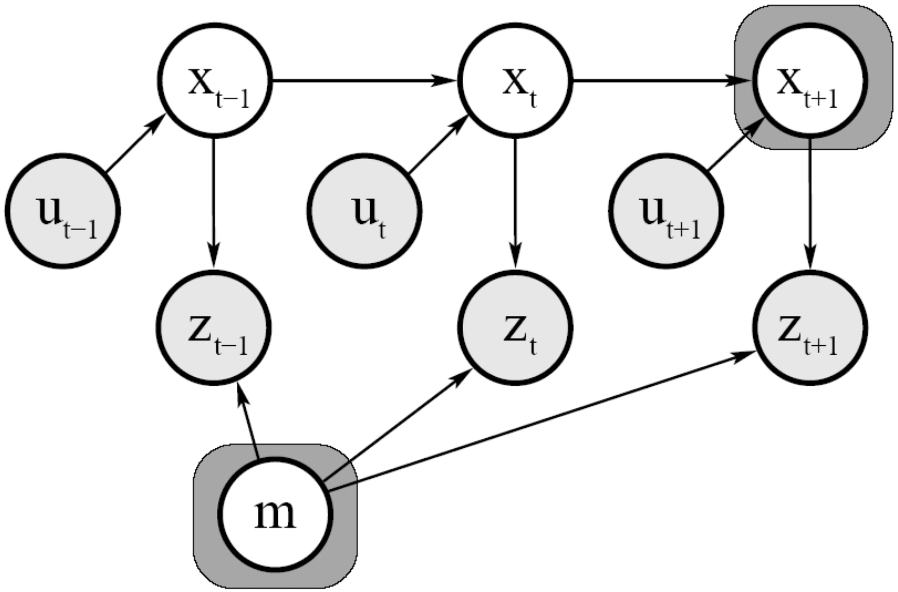
- **Correction**

$$bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$$

EKF for Online SLAM

- We consider here the Kalman filter as a solution to the online SLAM problem

$$p(x_t, m \mid z_{1:t}, u_{1:t})$$



Extended Kalman Filter Algorithm

Extended_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\ H_t^T (H_t\ \bar{\Sigma}_t\ H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t\ H_t)\ \bar{\Sigma}_t$$
$$return\ \mu_t, \Sigma_t$$

EKF SLAM

- Application of the EKF to SLAM
- Estimate robot's pose and locations of landmarks in the environment
- Assumption: known correspondences
- State space (for the 2D plane) is

$$x_t = (\underbrace{x, y, \theta}_{\text{robot's pose}} , \underbrace{m_{1,x}, m_{1,y}}_{\text{landmark 1}}, \ldots , \underbrace{m_{n,x}, m_{n,y}}_{\text{landmark n}})^T$$

EKF SLAM: State Representation

- Map with *n* landmarks: (3+2*n*)-dimensional Gaussian
- Belief is represented by

$$
\underbrace{\begin{pmatrix} x \\ y \\ \theta \\ m_{1,x} \\ m_{1,y} \\ \vdots \\ m_{n,x} \\ m_{n,y} \end{pmatrix}}_{\mu}
\quad
\underbrace{\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xm_{1,x}} & \sigma_{xm_{1,y}} & \cdots & \sigma_{xm_{n,x}} & \sigma_{xm_{n,y}} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{ym_{1,x}} & \sigma_{ym_{1,y}} & \cdots & \sigma_{m_{n,x}} & \sigma_{m_{n,y}} \\
\sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta m_{1,x}} & \sigma_{\theta m_{1,y}} & \cdots & \sigma_{\theta m_{n,x}} & \sigma_{\theta m_{n,y}} \\
\sigma_{m_{1,x}x} & \sigma_{m_{1,x}y} & \sigma_{\theta} & \sigma_{m_{1,x}m_{1,x}} & \sigma_{m_{1,x}m_{1,y}} & \cdots & \sigma_{m_{1,x}m_{n,x}} & \sigma_{m_{1,x}m_{n,y}} \\
\sigma_{m_{1,y}x} & \sigma_{m_{1,y}y} & \sigma_{\theta} & \sigma_{m_{1,y}m_{1,x}} & \sigma_{m_{1,y}m_{1,y}} & \cdots & \sigma_{m_{1,y}m_{n,x}} & \sigma_{m_{1,y}m_{n,y}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{m_{n,x}x} & \sigma_{m_{n,x}y} & \sigma_{\theta} & \sigma_{m_{n,x}m_{1,x}} & \sigma_{m_{n,x}m_{1,y}} & \cdots & \sigma_{m_{n,x}m_{n,x}} & \sigma_{m_{n,x}m_{n,y}} \\
\sigma_{m_{n,y}x} & \sigma_{m_{n,y}y} & \sigma_{\theta} & \sigma_{m_{n,y}m_{1,x}} & \sigma_{m_{n,y}m_{1,y}} & \cdots & \sigma_{m_{n,y}m_{n,x}} & \sigma_{m_{n,y}m_{n,y}}
\end{pmatrix}}_{\Sigma}
$$

EKF SLAM: State Representation

- More compactly

$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: State Representation

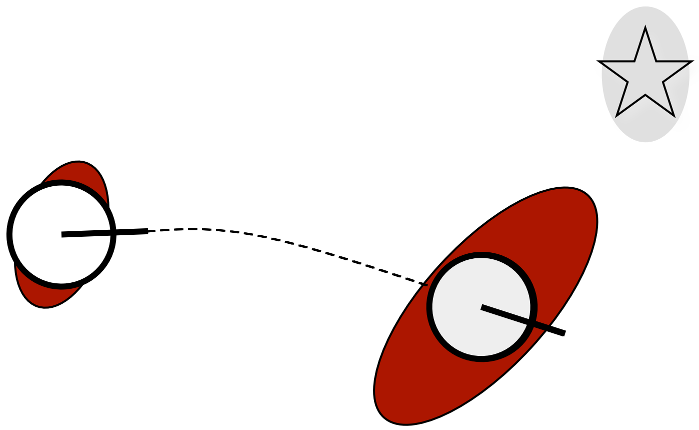
- Even more compactly (note: $x_R \rightarrow x$)

$$\underbrace{\begin{pmatrix} x \\ m \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Filter Cycle

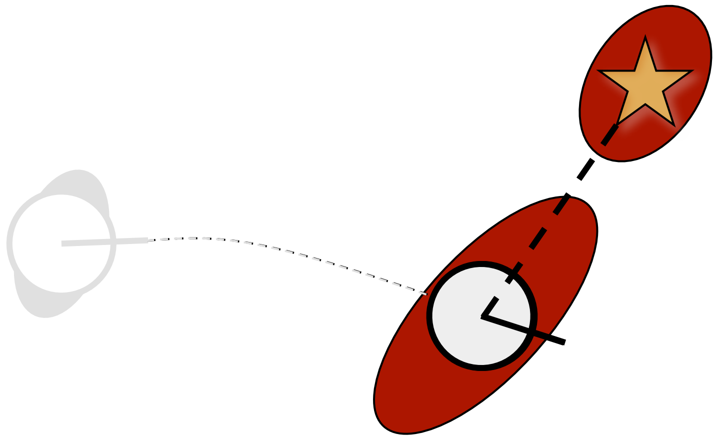
1. State prediction
2. Measurement prediction
3. Measurement
4. Data association
5. Update

EKF SLAM: State Prediction



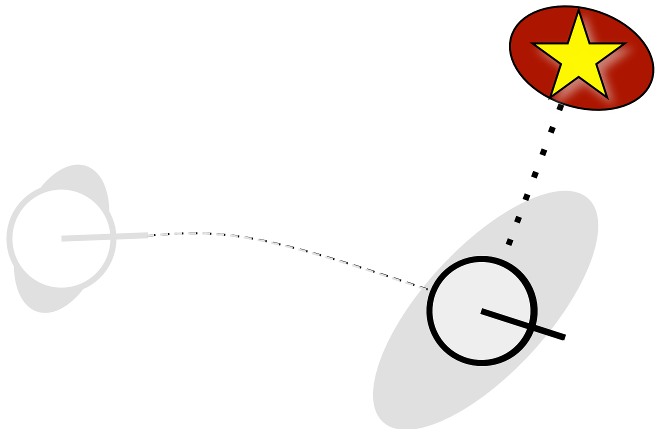
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Measurement Prediction



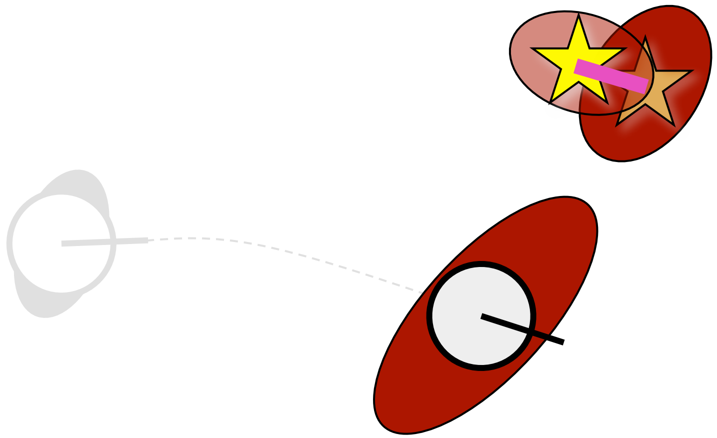
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Obtained Measurement



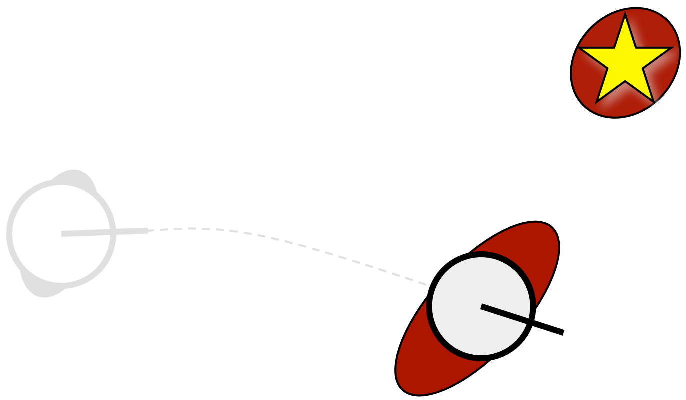
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Data Association and Difference Between h(x) and z



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Update Step



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Concrete Example

Setup

- Robot moves in the 2D plane
- Velocity-based motion model
- Robot observes point landmarks
- Range-bearing sensor
- Known data association
- Known number of landmarks

Initialization

- Robot starts in its own reference frame (all landmarks unknown)
- 2N+3 dimensions

$$\mu_0 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^T$$

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \infty & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \infty \end{pmatrix}$$

Extended Kalman Filter Algorithm

Extended_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$$
$$\text{return } \mu_t, \Sigma_t$$

Prediction Step (Motion)

- Goal: Update state space based on the robot's motion
- Robot motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g_{x,y,\theta}(u_t,(x,y,\theta)^T)}$$

- How to map that to the 2N+3 dim space?

Update the State Space

- From the motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}$$

- to the 2N+3 dimensional space

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \\ \vdots \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0\dots0 \\ 0 & 1 & 0 & 0\dots0 \\ 0 & 0 & 1 & \underbrace{0\dots0}_{2N cols} \end{pmatrix}^T}_{F_x^T} \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}}_{g(u_t,x_t)}$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$\text{return } \mu_t, \Sigma_t$$

Update Covariance

- The function g only affects the robot's motion and not the landmarks

Jacobian of the motion (3x3)

$$G_t = \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix}$$

Identity (2N x 2N)

Jacobian of the Motion

$$G_t^x = \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}\right]$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial (x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial (x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

This Leads to the Update

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: $\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$ **Apply & DONE**

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

$$
\begin{aligned}
\bar{\Sigma}_t &= G_t \Sigma_{t-1} G_t^T + R_t \\
&= \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix} \begin{pmatrix} (G_t^x)^T & 0 \\ 0 & I \end{pmatrix} + R_t \\
&= \begin{pmatrix} G_t^x \Sigma_{xx} (G_t^x)^T & G_t^x \Sigma_{xm} \\ (G_t^x \Sigma_{xm})^T & \Sigma_{mm} \end{pmatrix} + R_t
\end{aligned}
$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t}$$ **DONE**

$$K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$$
$$\textit{return}\ \mu_t, \Sigma_t$$

EKF SLAM:Prediction Step

EKF_SLAM_Prediction$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, R_t)$**:**

2: $$F_x = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \end{pmatrix}$$

3: $$\bar{\mu}_t = \mu_{t-1} + F_x^T \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

4: $$G_t = I + F_x^T \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} F_x$$

5: $$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + \underbrace{F_x^T \, R_t^x \, F_x}_{R_t}$$

Extended Kalman Filter Algorithm

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$

2: $\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$ **DONE**

3: $\cancel{\bar{\Sigma}_t = G_t\,\Sigma_{t-1}\,G_t^T + R_t}$ **Apply & DONE**

4: $K_t = \bar{\Sigma}_t\,H_t^T(H_t\,\bar{\Sigma}_t\,H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t\,H_t)\,\bar{\Sigma}_t$

7: $return\ \mu_t, \Sigma_t$

EKF SLAM: Correction Step

- Known data association
- $c_t^i = j$: i-th measurement at time t observes the landmark with index j
- Initialize landmark if unobserved
- Compute the expected observation
- Compute the Jacobian of h
- Proceed with computing the Kalman gain

Range-Bearing Observation

- Range-Bearing observation $z_t^i = (r_t^i, \phi_t^i)^T$
- If landmark has not been observed

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \ \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \ \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

observed location of landmark j	estimated robot's location	relative measurement

Expected Observation

- Compute expected observation according to the current estimate

$$
\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \\
&= h(\bar{\mu}_t)
\end{aligned}
$$

Jacobian for the Observation

- Based on

$$
\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}
\end{aligned}
$$

- Compute the Jacobian

$$
{}^{\text{low}}H_t^i = \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t}
$$

low-dim space $(x, y, \theta, m_{j,x}, m_{j,y})$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \mathrm{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$\begin{aligned} {}^{\mathrm{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \begin{pmatrix} \frac{\partial \sqrt{q}}{\partial x} & \frac{\partial \sqrt{q}}{\partial y} & \dots \\ \frac{\partial \mathrm{atan2}(\dots)}{\partial x} & \frac{\partial \mathrm{atan2}(\dots)}{\partial y} & \dots \end{pmatrix} \end{aligned}$$

The First Component

- Based on

$$
\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}
\end{aligned}
$$

- We obtain (by applying the chain rule)

$$
\begin{aligned}
\frac{\partial \sqrt{q}}{\partial x} &= \frac{1}{2} \frac{1}{\sqrt{q}} \, 2 \, \delta_x \, (-1) \\
&= \frac{1}{q} (-\sqrt{q} \delta_x)
\end{aligned}
$$

Jacobian for the Observation

- Based on

$$\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \mathrm{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}
\end{aligned}$$

- Compute the Jacobian

$$\begin{aligned}
{}^{\text{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\
&= \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}
\end{aligned}$$

Jacobian for the Observation

- Use the computed Jacobian

$$
{}^{\text{low}}H_t^i = \frac{1}{q}\begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}
$$

- map it to the high dimensional space

$$
H_t^i = {}^{\text{low}}H_t^i \, F_{x,j}
$$

$$
F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}
$$

Next Steps as Specified...

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t}$$ **DONE**

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$\text{return } \mu_t, \Sigma_t$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t}$$ **DONE**

$$\cancel{K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}}$$ **Apply & DONE**

$$\cancel{\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))}$$ **Apply & DONE**

$$\cancel{\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t}$$ **Apply & DONE**

$$return\ \mu_t, \Sigma_t$$

EKF SLAM – Correction (1/2)

EKF_SLAM_Correction

$$Q_t = \begin{pmatrix} {\sigma_r}^2 & 0 \\ 0 & {\sigma_\phi}^2 \end{pmatrix}$$

for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ *do*

$\quad j = c_t^i$

$\quad$ *if landmark* j *never seen before*

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

$\quad$ *endif*

$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$

$\quad q = \delta^T \delta$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

EKF SLAM – Correction (2/2)

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}$$

$$H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & +\delta_x \end{pmatrix} F_{x,j}$$

$$K_t^i = \bar{\Sigma}_t \; H_t^{iT} (H_t^i \; \bar{\Sigma}_t \; H_t^{iT} + Q_t)^{-1}$$

$$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$$

$$\bar{\Sigma}_t = (I - K_t^i \; H_t^i) \; \bar{\Sigma}_t$$

endfor

$$\mu_t = \bar{\mu}_t$$

$$\Sigma_t = \bar{\Sigma}_t$$

return μ_t, Σ_t

Implementation Notes

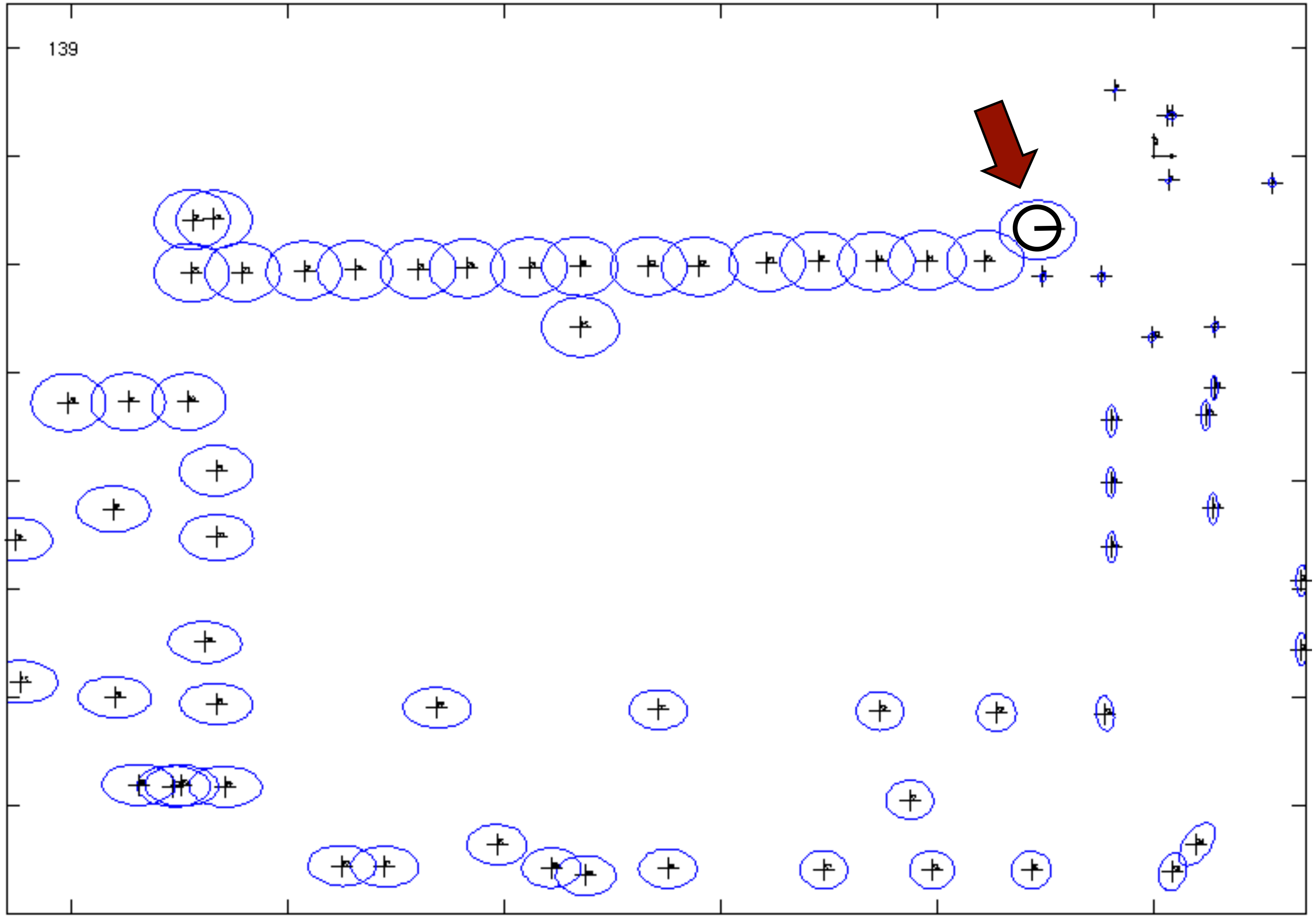
- Measurement update in a single step requires only one full belief update
- Always normalize the angular components
- You may not need to create the F matrices explicitly (e.g., in Octave)

Done!

Loop Closing

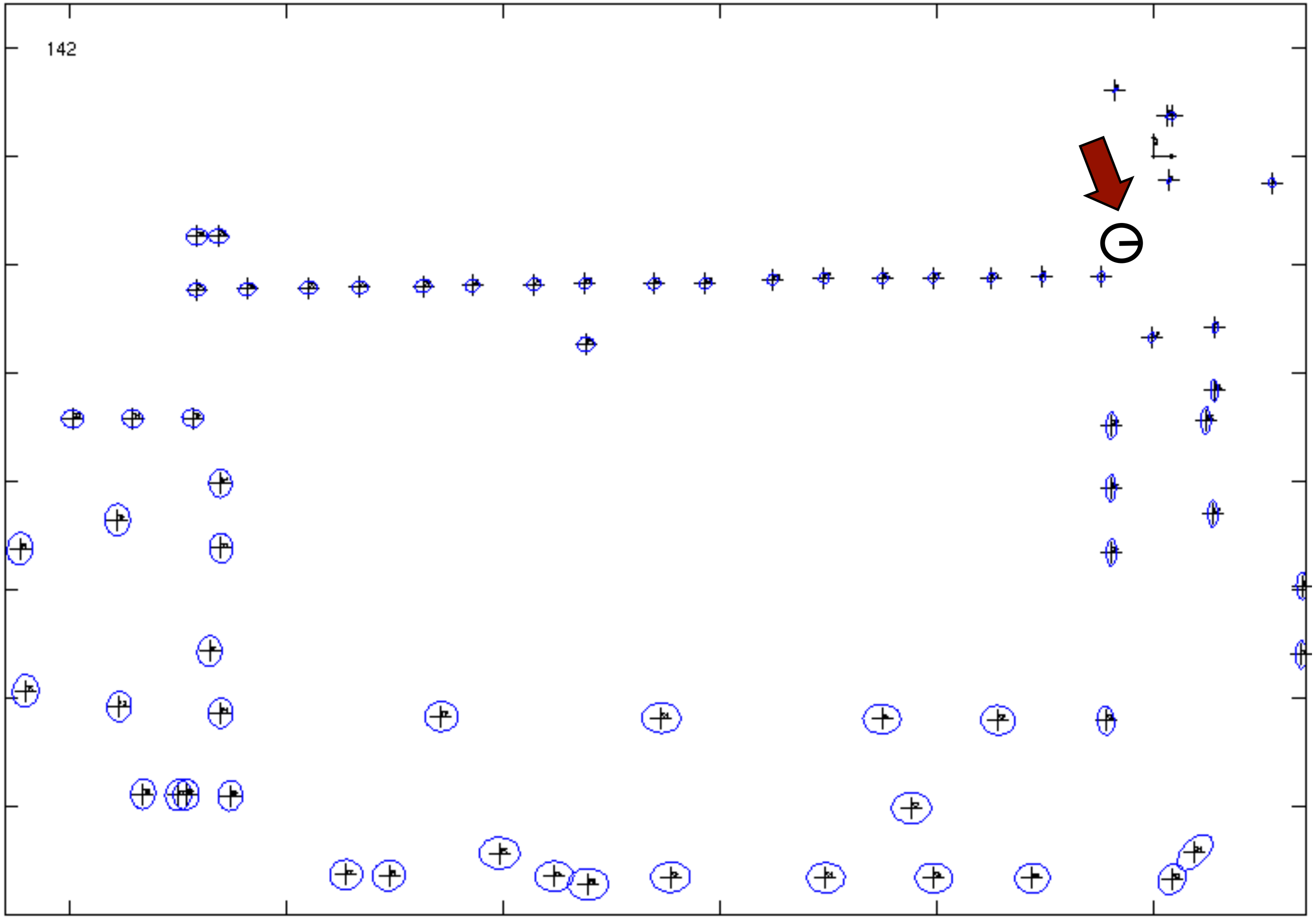
- Loop closing means recognizing an already mapped area
- Data association under
  - high ambiguity
  - possible environment symmetries
- Uncertainties **collapse** after a loop closure (whether the closure was correct or not)

Before the Loop Closure



Courtesy of K. Arras

After the Loop Closure



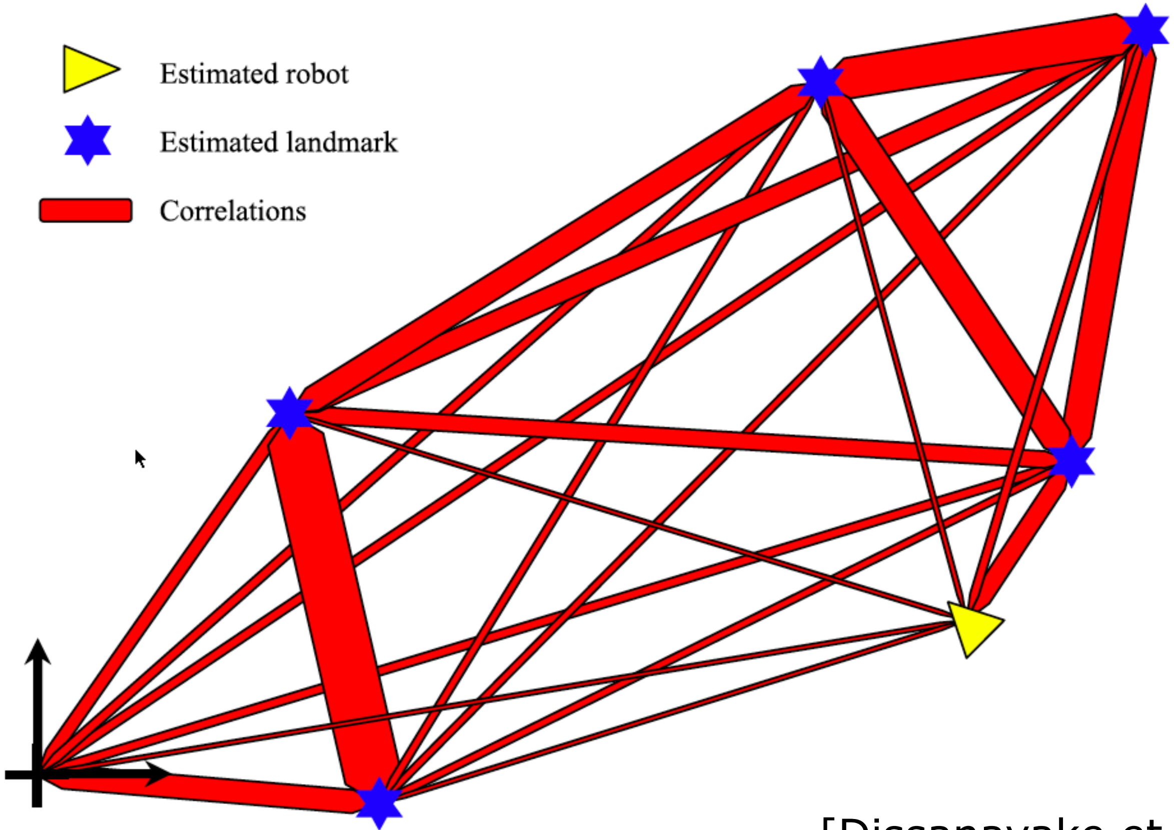
Courtesy of K. Arras

Loop Closures in SLAM

- Loop closing **reduces** the uncertainty in robot and landmark estimates
- This can be exploited when exploring an environment for the sake of better (e.g. more accurate) maps
- **Wrong loop closures lead to filter divergence**

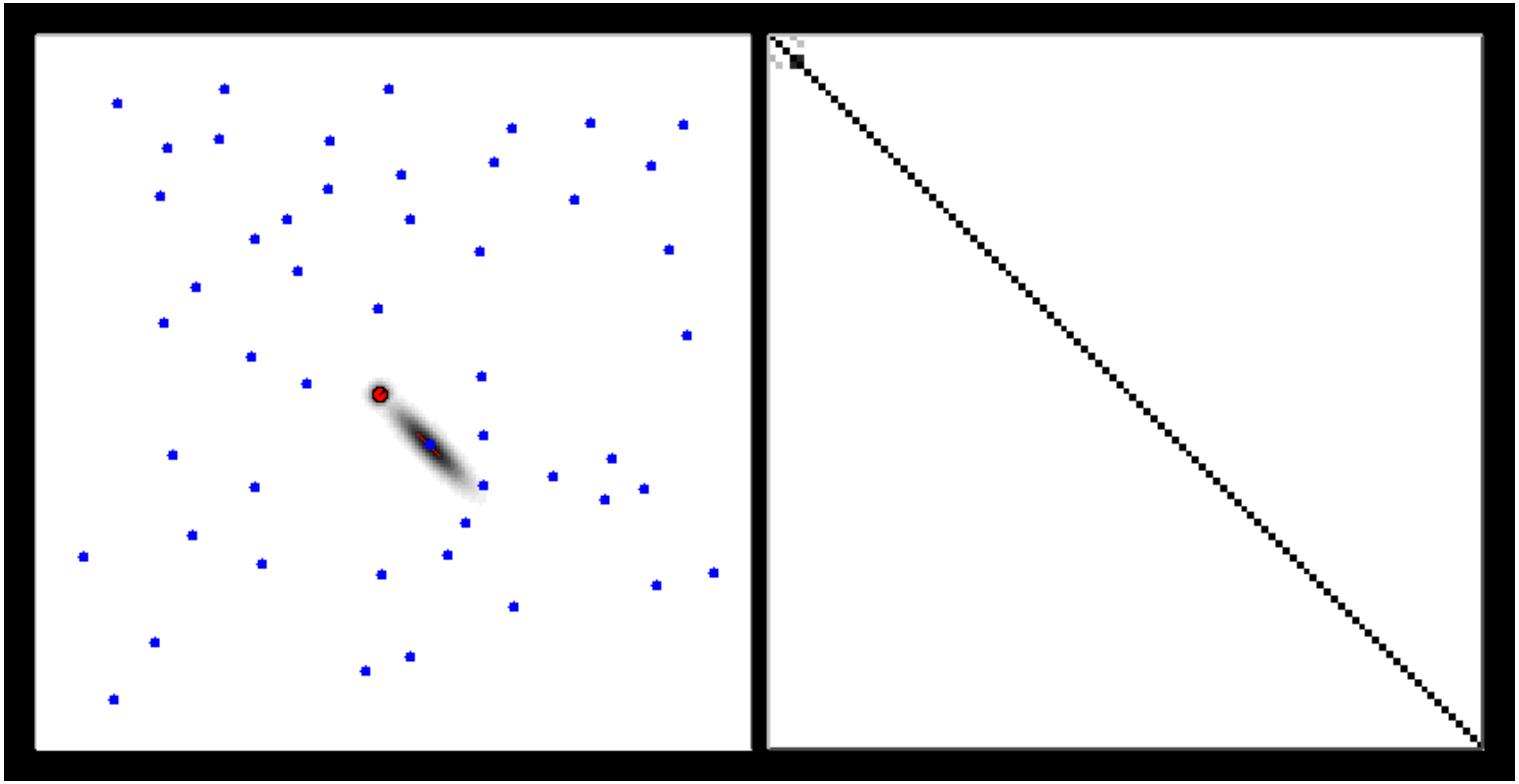
EKF SLAM Correlations

- In the limit, the landmark estimates become **fully correlated**



[Dissanayake et al., 2001]

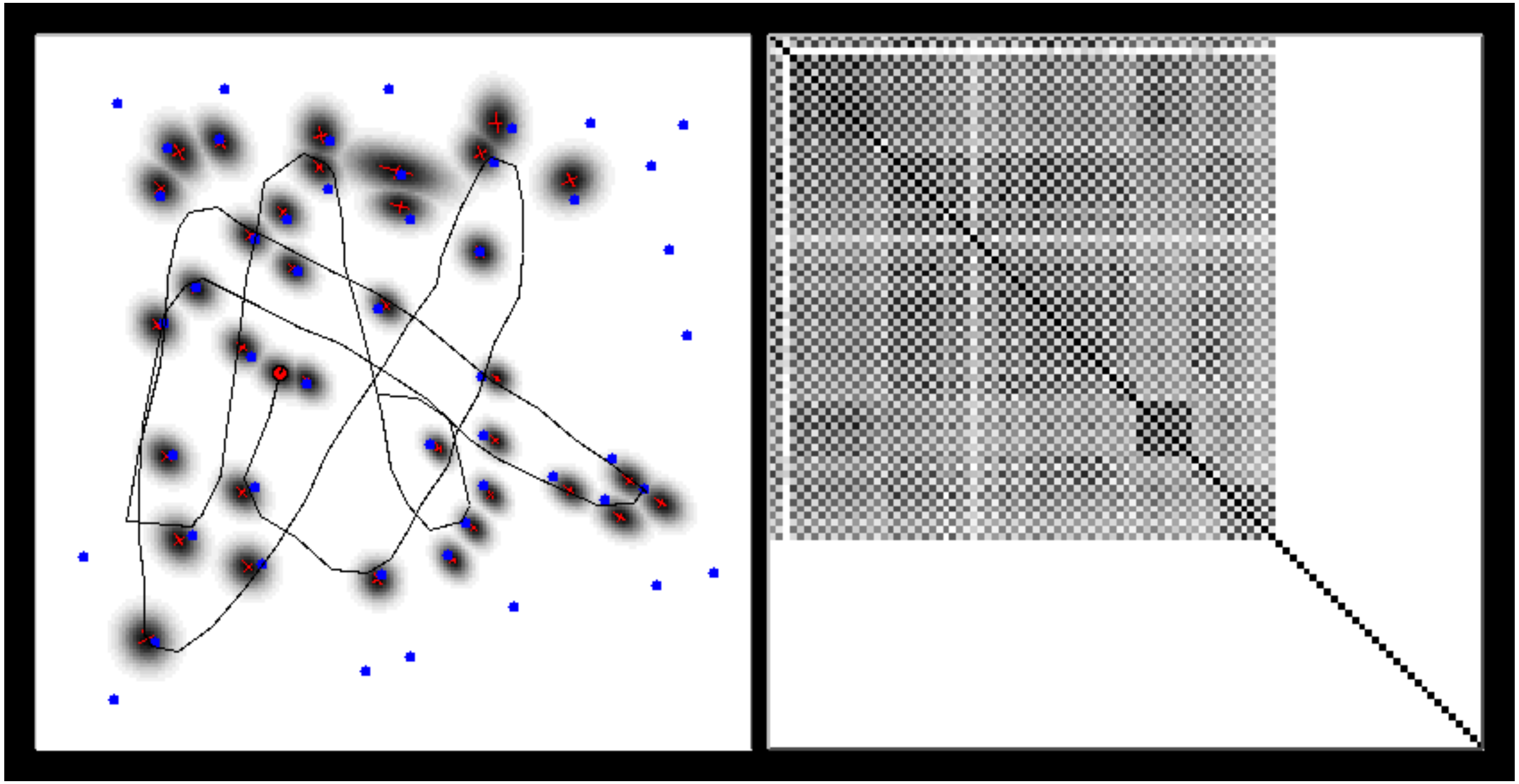
EKF SLAM Correlations



Map Correlation matrix

Courtesy of M. Montemerlo

EKF SLAM Correlations

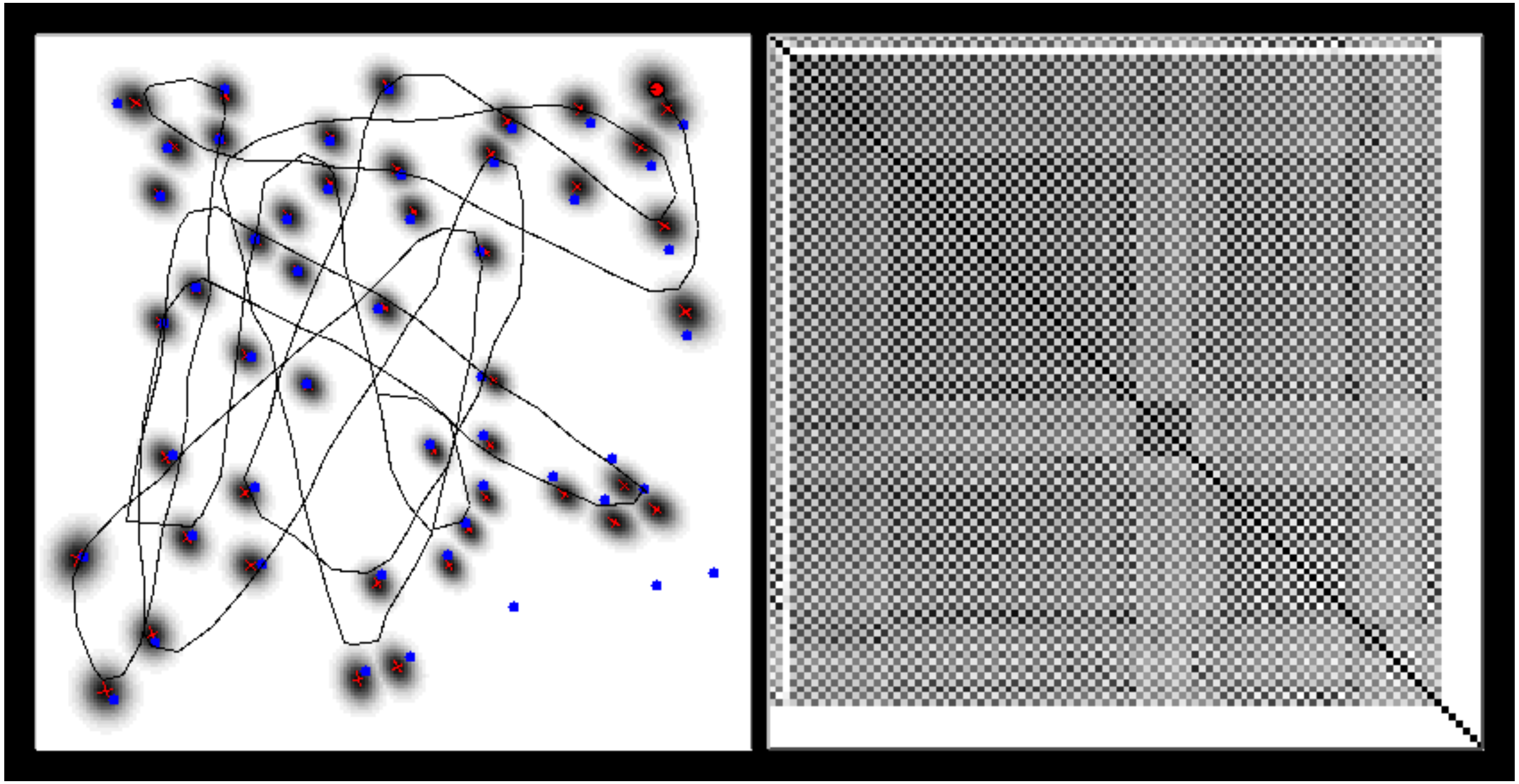


Map

Correlation matrix

Courtesy of M. Montemerlo

EKF SLAM Correlations



Map Correlation matrix

Courtesy of M. Montemerlo

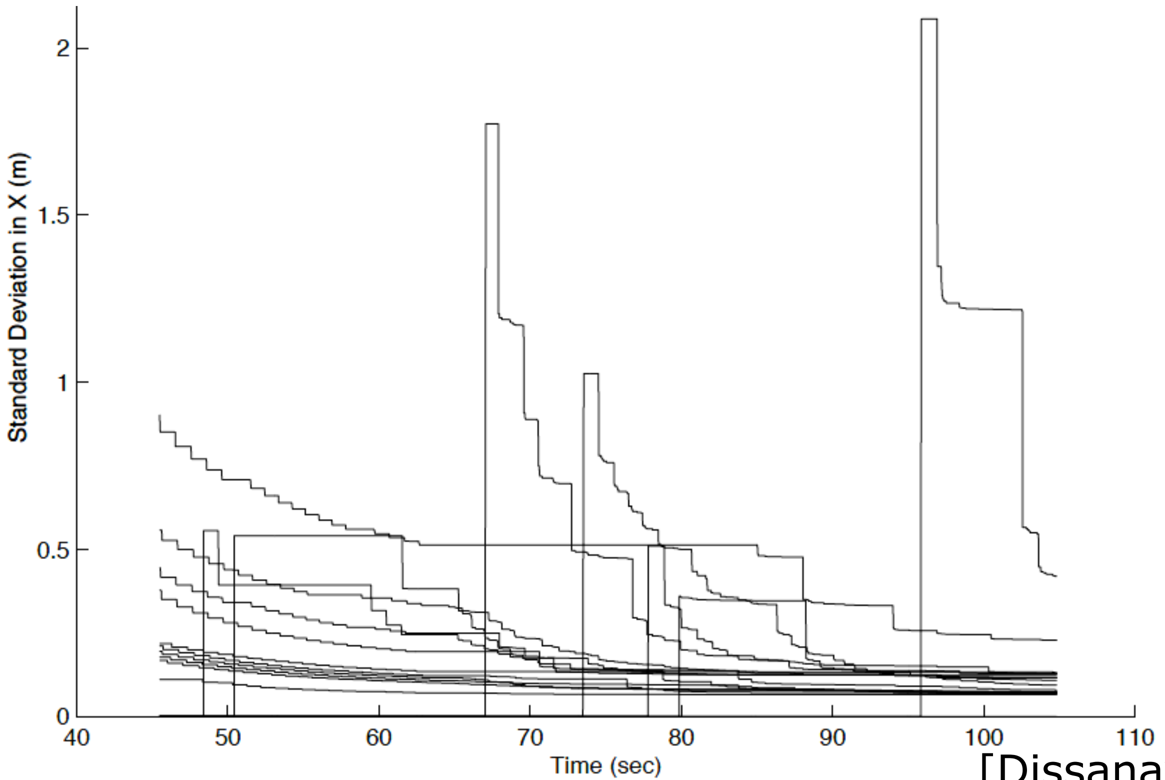
EKF SLAM Correlations

- The correlation between the robot's pose and the landmarks **cannot** be ignored
- Assuming independence generates too optimistic estimates of the uncertainty

[Castellanos et al., 1997]

EKF SLAM Uncertainties

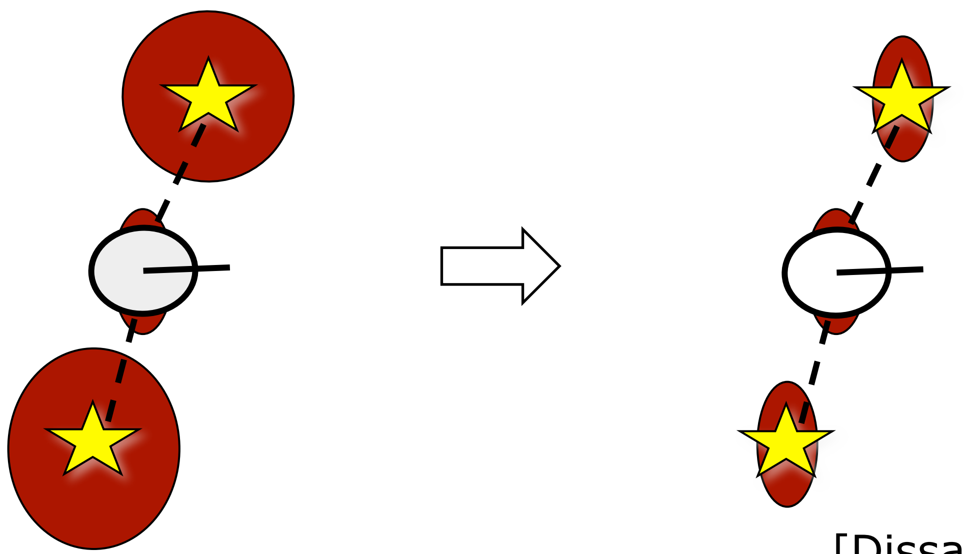
- The **determinant** of any sub-matrix of the map covariance matrix **decreases monotonically**
- New landmarks are initialized with **maximum uncertainty**



[Dissanayake et al., 2001]

EKF SLAM in the Limit

- In the limit, the covariance associated with any single landmark location estimate is determined only by the initial covariance in the vehicle location estimate.



[Dissanayake et al., 2001]

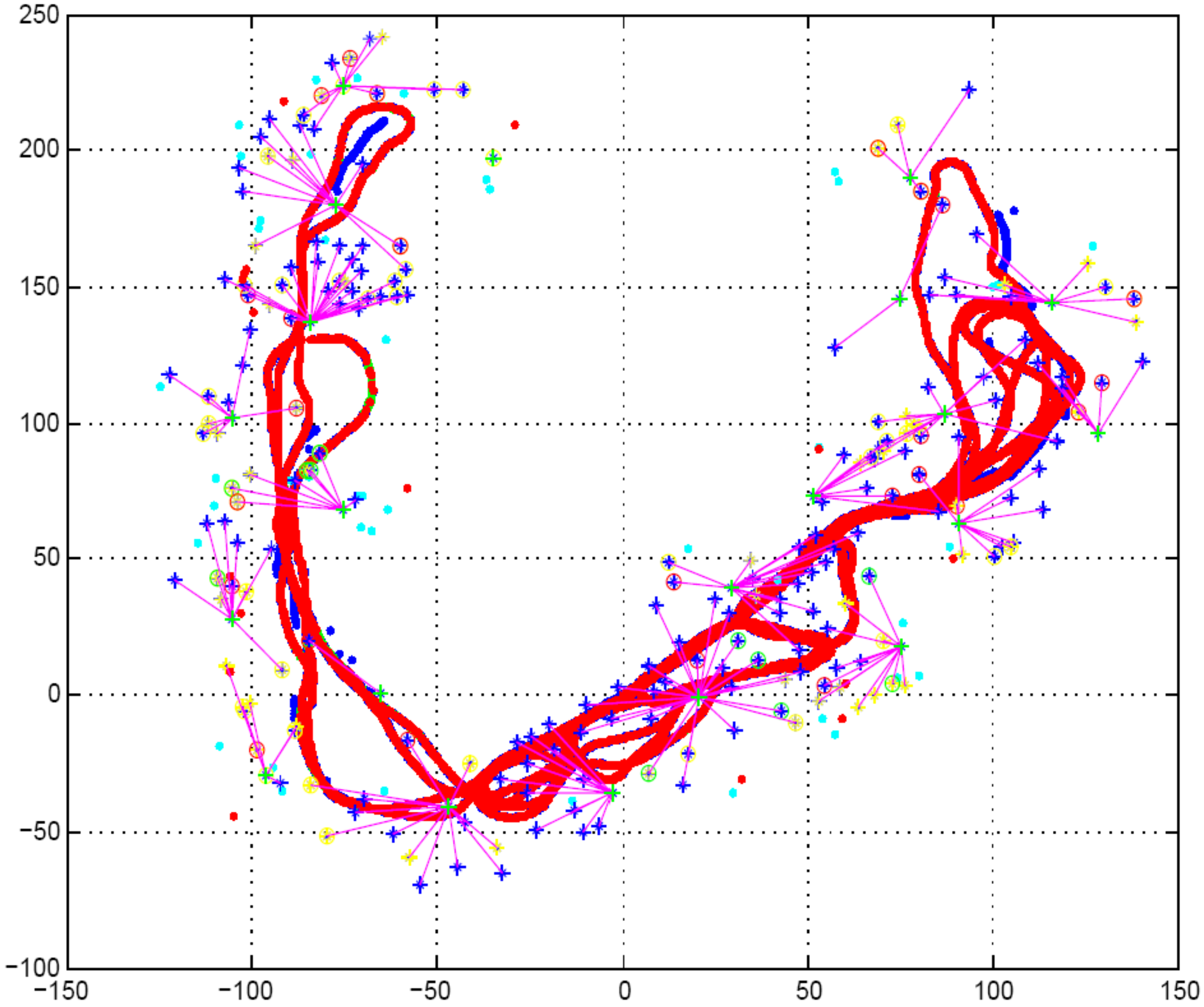
Example: Victoria Park Dataset

Courtesy of E. Nebot

Victoria Park: Data Acquisition

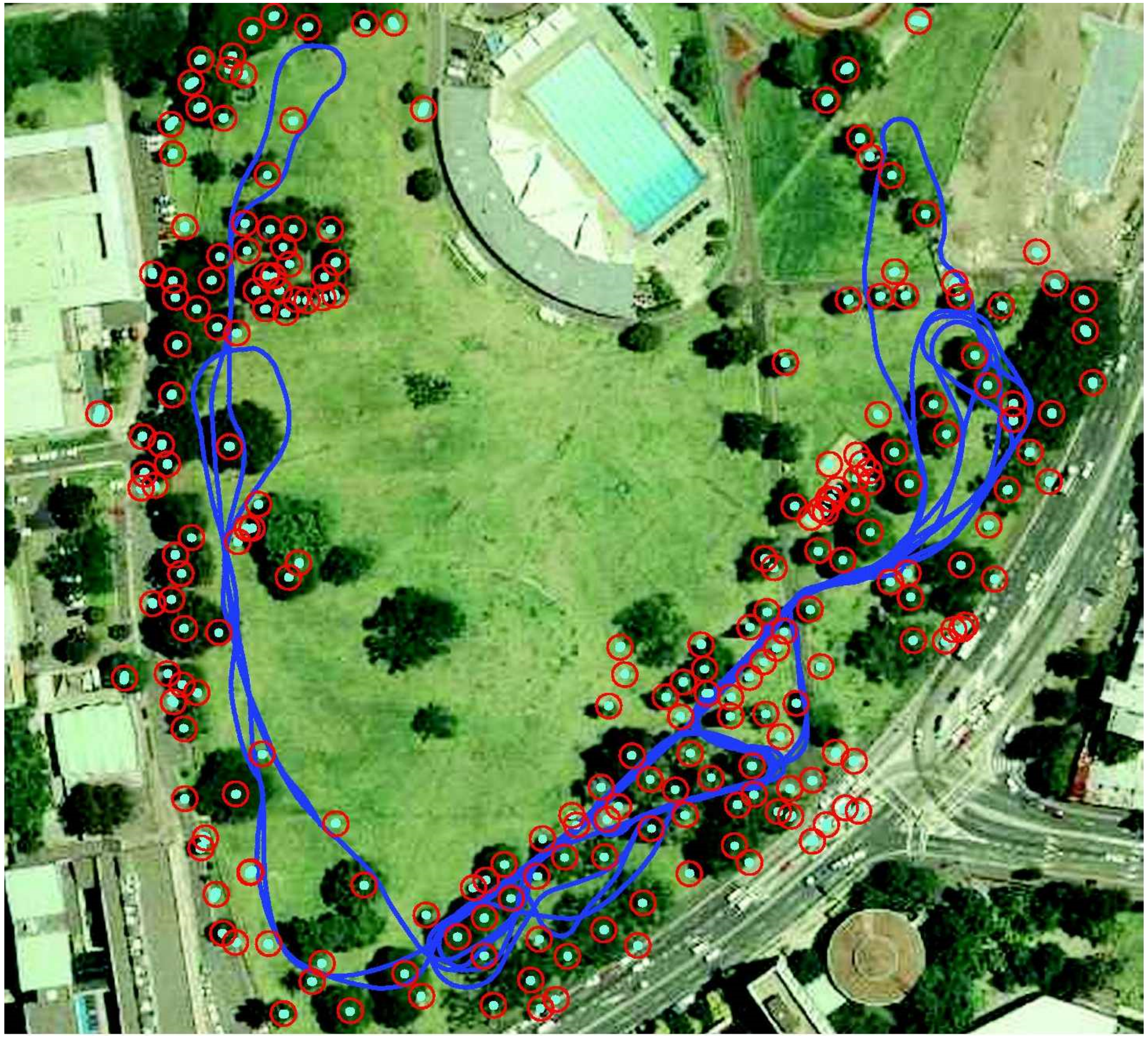
Courtesy of E. Nebot

Victoria Park: EKF Estimate



Courtesy of E. Nebot

Victoria Park: Landmarks

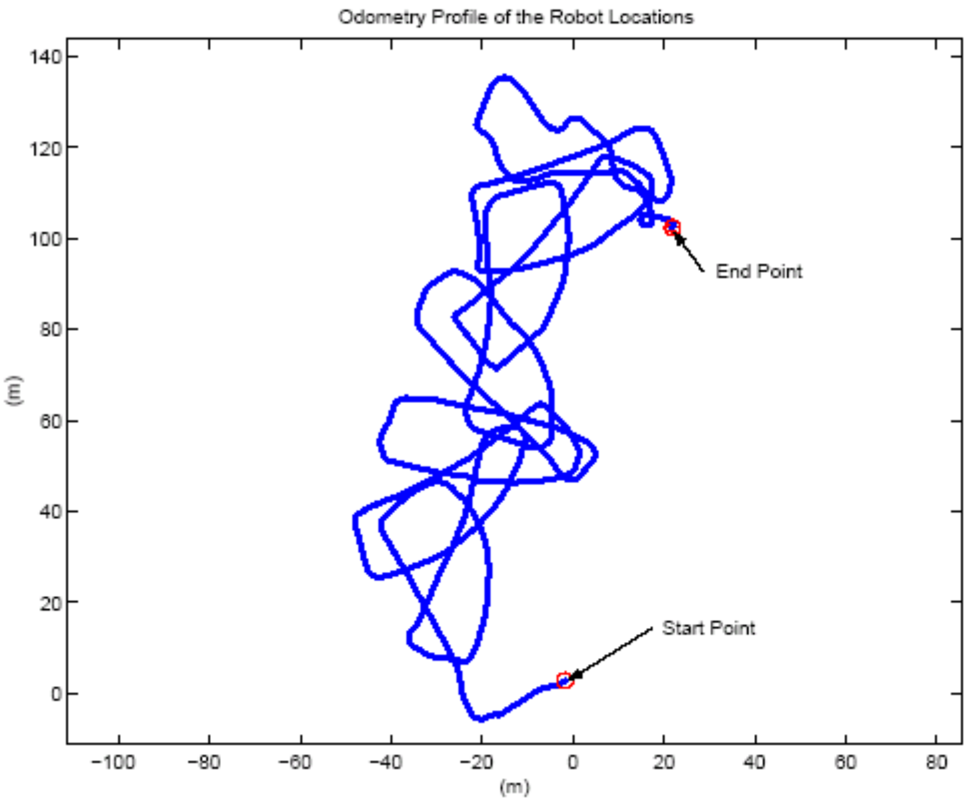


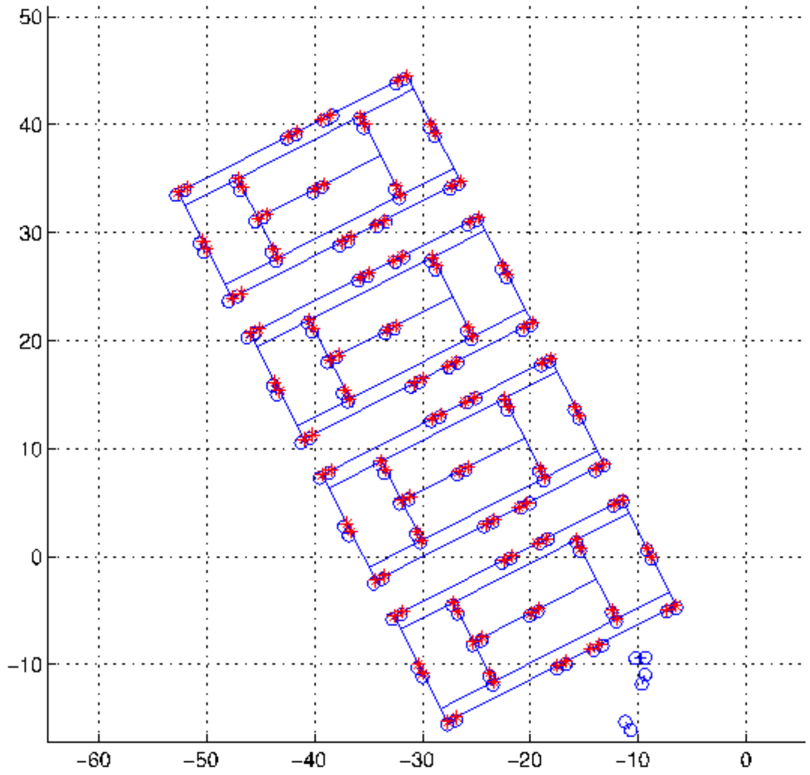
Courtesy of E. Nebot

Example: Tennis Court Dataset

Courtesy of J. Leonard and M. Walter

EKF SLAM on a Tennis Court





odometry

estimated trajectory

Courtesy of J. Leonard and M. Walter

EKF SLAM Complexity

- Cubic complexity depends only on the measurement dimensionality
- Cost per step: dominated by the number of landmarks: $O(n^2)$
- Memory consumption: $O(n^2)$
- The EKF becomes computationally intractable for large maps!

EKF SLAM Summary

- The first SLAM solution
- Convergence proof for the linear Gaussian case
- Can diverge if non-linearities are large (and the reality is non-linear...)
- Can deal only with a single mode
- Successful in medium-scale scenes
- Approximations exists to reduce the computational complexity

Literature

EKF SLAM

- Thrun et al.: “Probabilistic Robotics”, Chapter 10