Robot Mapping

A Short Introduction to the Bayes Filter and Related Models

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State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x \mid z, u)$$

Recursive Bayes Filter 1

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

Definition of the belief

Recursive Bayes Filter 2

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \underline{\eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\; p(x_t \mid z_{1:t-1}, u_{1:t})}
\end{aligned}
$$

Bayes' rule

Recursive Bayes Filter 3

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, \underline{p(z_t \mid x_t)}\, p(x_t \mid z_{1:t-1}, u_{1:t})
\end{aligned}
$$

Markov assumption

Recursive Bayes Filter 4

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \underline{\int_{x_{t-1}} p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})} \\
&\qquad \underline{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}}
\end{aligned}
$$

Law of total probability

Recursive Bayes Filter 5

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} \underline{p(x_t \mid x_{t-1}, u_t)}\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}
\end{aligned}
$$

Markov assumption

Recursive Bayes Filter 6

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:\underline{t-1}})\, dx_{t-1}
\end{aligned}
$$

Markov assumption

Recursive Bayes Filter 7

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, \underline{bel(x_{t-1})}\, dx_{t-1}
\end{aligned}
$$

Recursive term

Prediction and Correction Step

- Bayes filter can be written as a two step process
- **Prediction step**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\, dx_{t-1}$$

- **Correction step**

$$bel(x_t) = \eta\, p(z_t \mid x_t)\ \overline{bel}(x_t)$$

Motion and Observation Model

- Prediction step

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\textbf{motion model}} \, bel(x_{t-1}) \, dx_{t-1}$$

- Correction step

$$bel(x_t) = \eta \, \underbrace{p(z_t \mid x_t)}_{\textbf{sensor or observation model}} \, \overline{bel}(x_t)$$

Different Realizations

- The Bayes filter is a **framework** for recursive state estimation
- There are **different realizations**
- **Different properties**
  - Linear vs. non-linear models for motion and observation models
  - Gaussian distributions only?
  - Parametric vs. non-parametric filters
  - …

In this Course

- **Kalman filter & friends**
  - Gaussians
  - Linear or linearized models

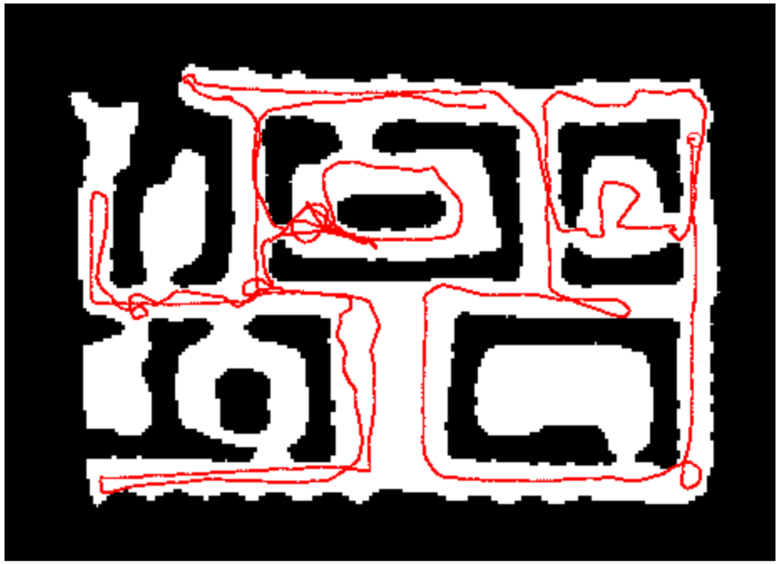
- **Particle filter**
  - Non-parametric
  - Arbitrary models (sampling required)

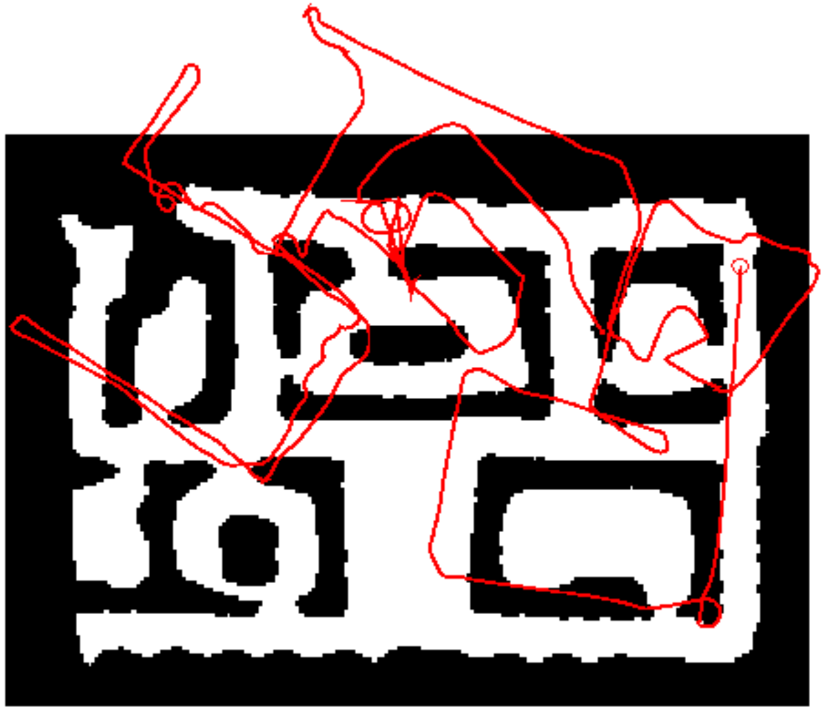
Motion Model

$$\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx_{t-1}$$

Robot Motion Models

- Robot motion is inherently uncertain
- How can we model this uncertainty?





Probabilistic Motion Models

- Specifies a posterior probability that action *u* carries the robot from *x* to *x′*.

$$p(x_t \mid u_t, x_{t-1})$$

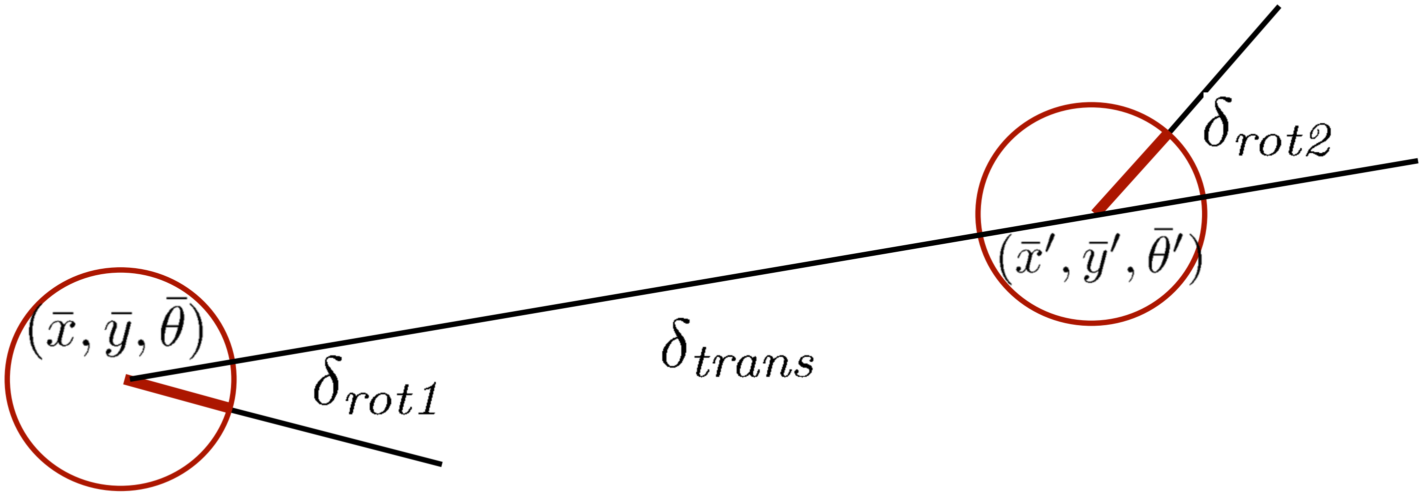
Typical Motion Models

- In practice, one often finds two types of motion models:
  - **Odometry-based**
  - **Velocity-based**
- Odometry-based models for systems that are equipped with wheel encoders
- Velocity-based when no wheel encoders are available

Odometry Model

- Robot moves from $(\bar{x}, \bar{y}, \bar{\theta})$ to $(\bar{x}', \bar{y}', \bar{\theta}')$
- Odometry information $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$

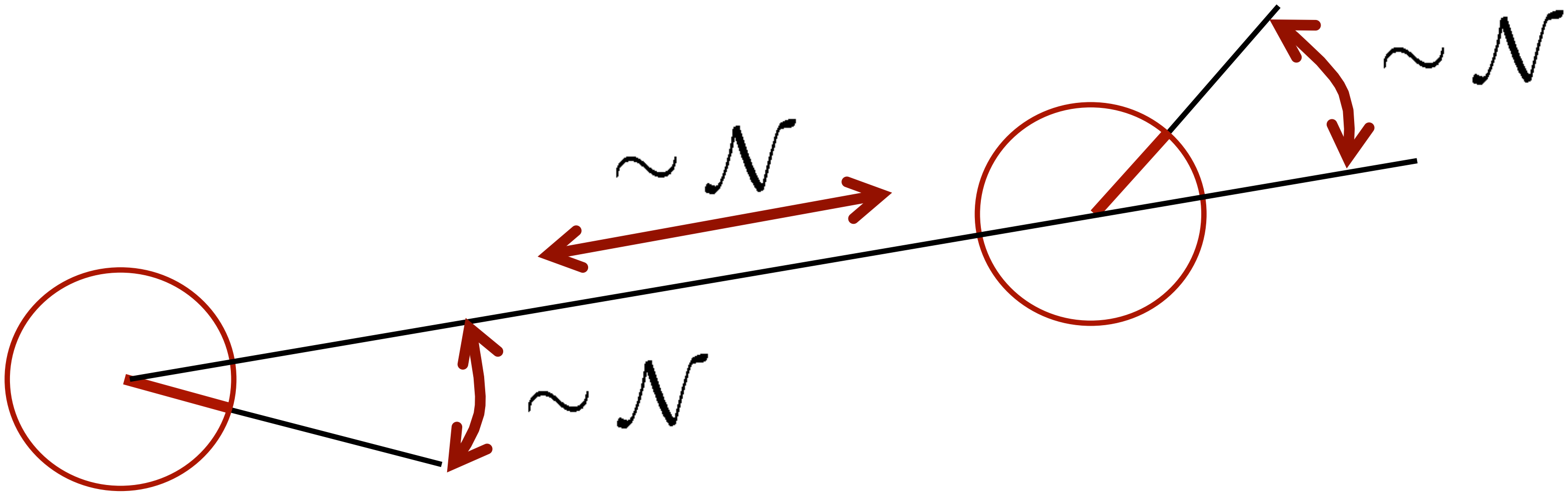
$$
\begin{aligned}
\delta_{trans} &= \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2} \\
\delta_{rot1} &= \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta} \\
\delta_{rot2} &= \bar{\theta}' - \bar{\theta} - \delta_{rot1}
\end{aligned}
$$



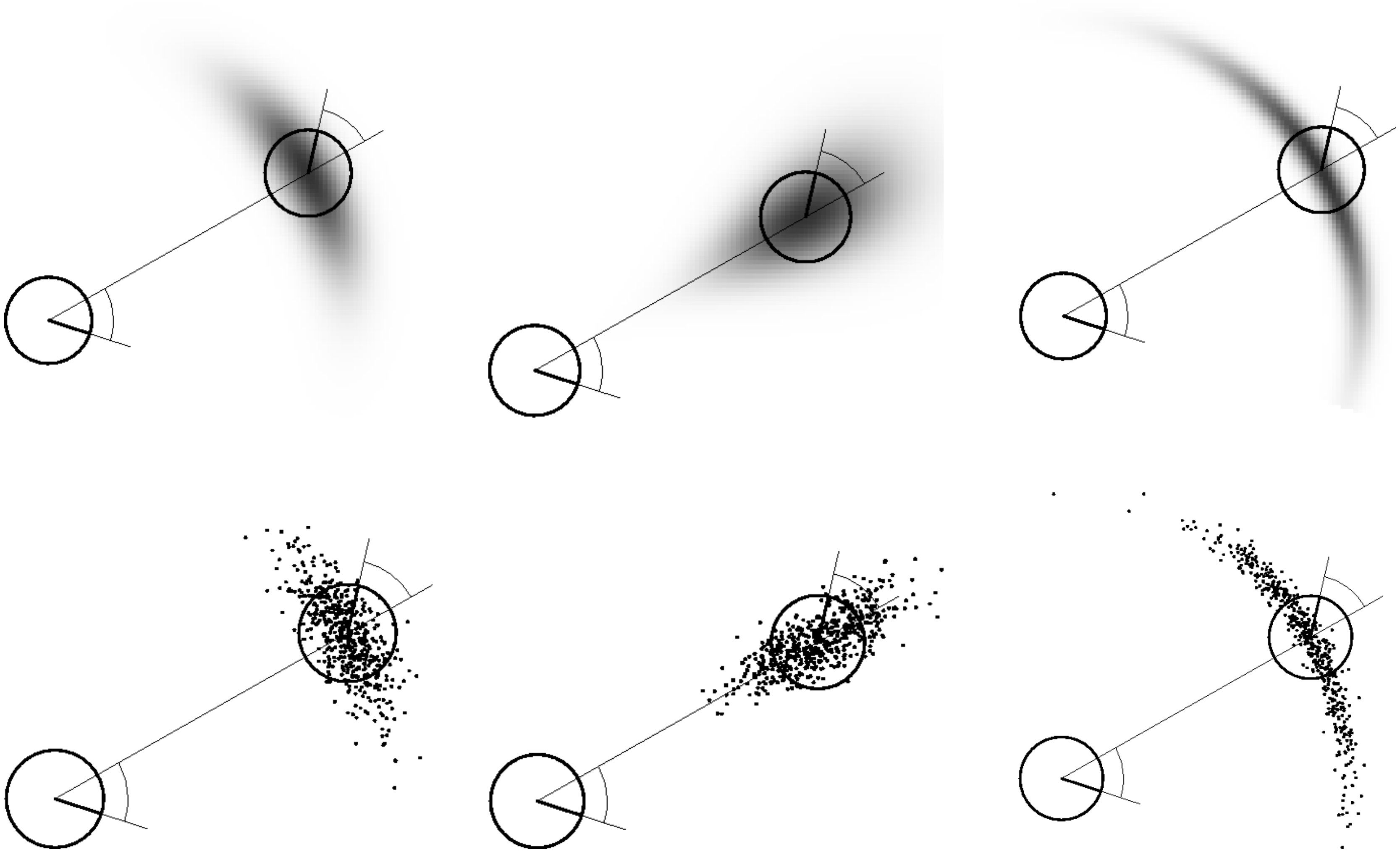
Probability Distribution

- Noise in odometry $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$
- Example: Gaussian noise

$$u \sim \mathcal{N}(0, \Sigma)$$

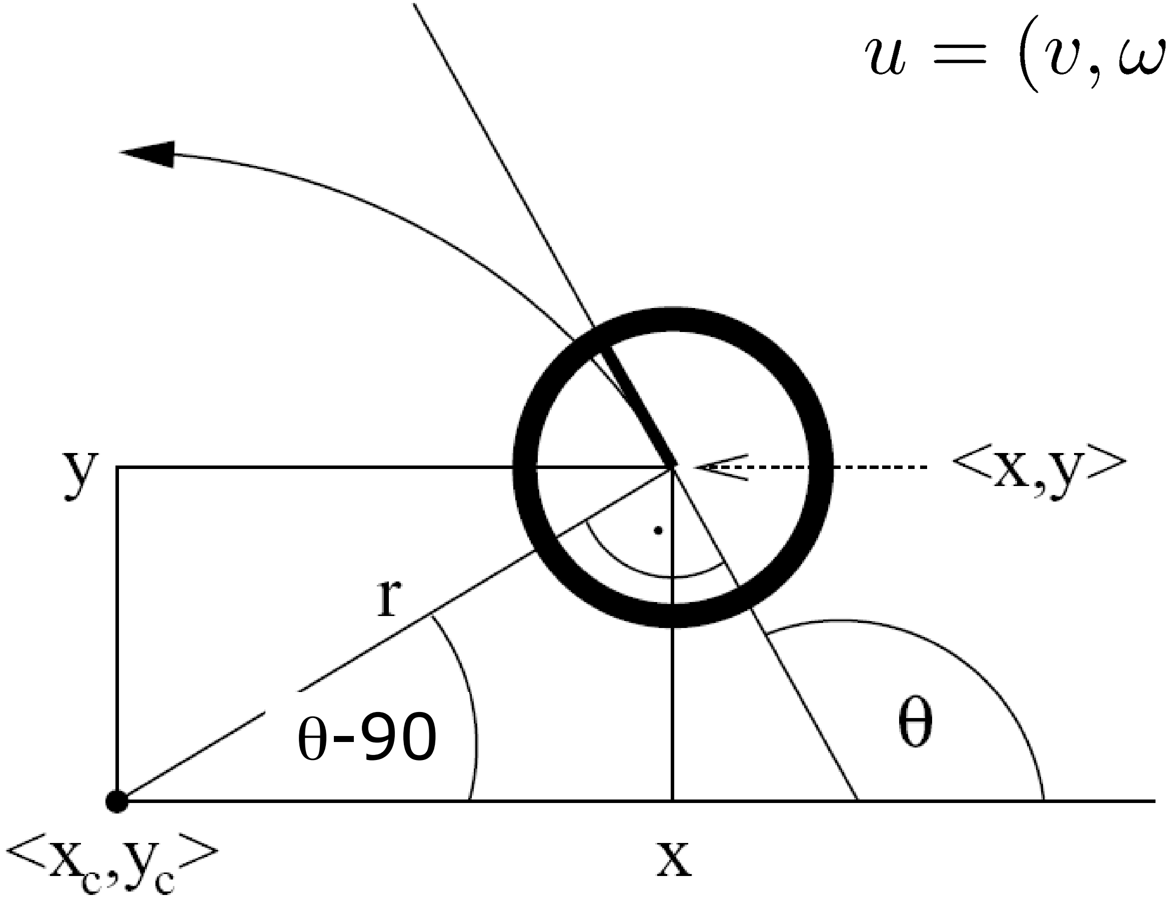


Examples (Odometry-Based)



Velocity-Based Model

$$u = (v, \omega)^T$$



Motion Equation

- Robot moves from (x, y, θ) to (x', y', θ')
- Velocity information $u = (v, \omega)$

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega}\sin\theta + \frac{v}{\omega}\sin(\theta + \omega\Delta t) \\ \frac{v}{\omega}\cos\theta - \frac{v}{\omega}\cos(\theta + \omega\Delta t) \\ \omega\Delta t \end{pmatrix}$$

Problem of the Velocity-Based Model

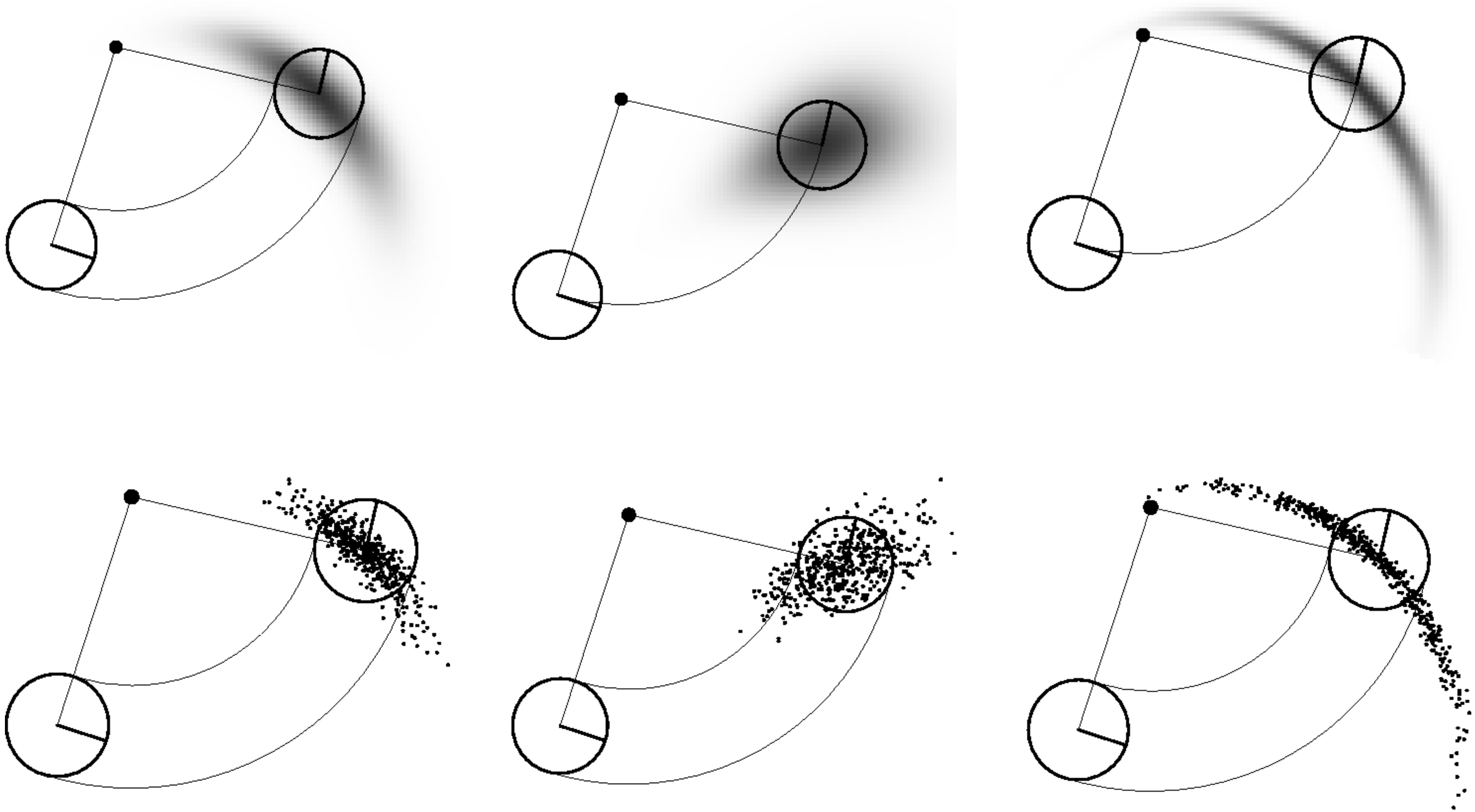
- Robot moves on a circle
- The circle constrains the final orientation
- **Fix:** introduce an additional noise term on the final orientation

Motion Including 3rd Parameter

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega}\sin\theta + \frac{v}{\omega}\sin(\theta + \omega\Delta t) \\ \frac{v}{\omega}\cos\theta - \frac{v}{\omega}\cos(\theta + \omega\Delta t) \\ \omega\Delta t + \underset{\uparrow}{\gamma}\Delta t \end{pmatrix}$$

Term to account for the final rotation

Examples (Velocity-Based)



Sensor Model

$$bel(x_t) = \eta \boxed{p(z_t \mid x_t)} \overline{bel}(x_{t-1})$$

Model for Laser Scanners

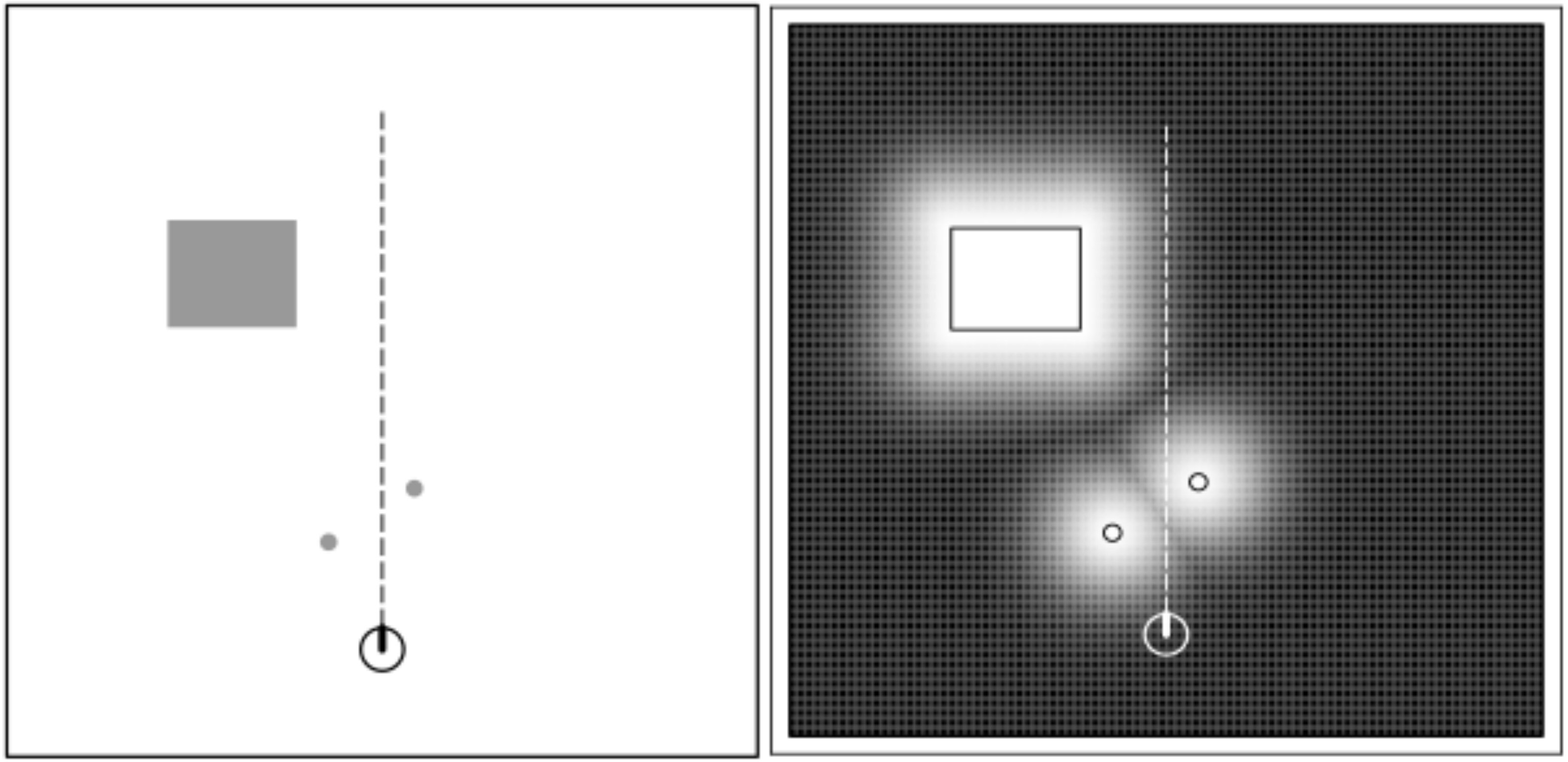
- Scan *z* consists of *K* measurements.

$$z_t = \{z_t^1, \dots, z_t^k\}$$

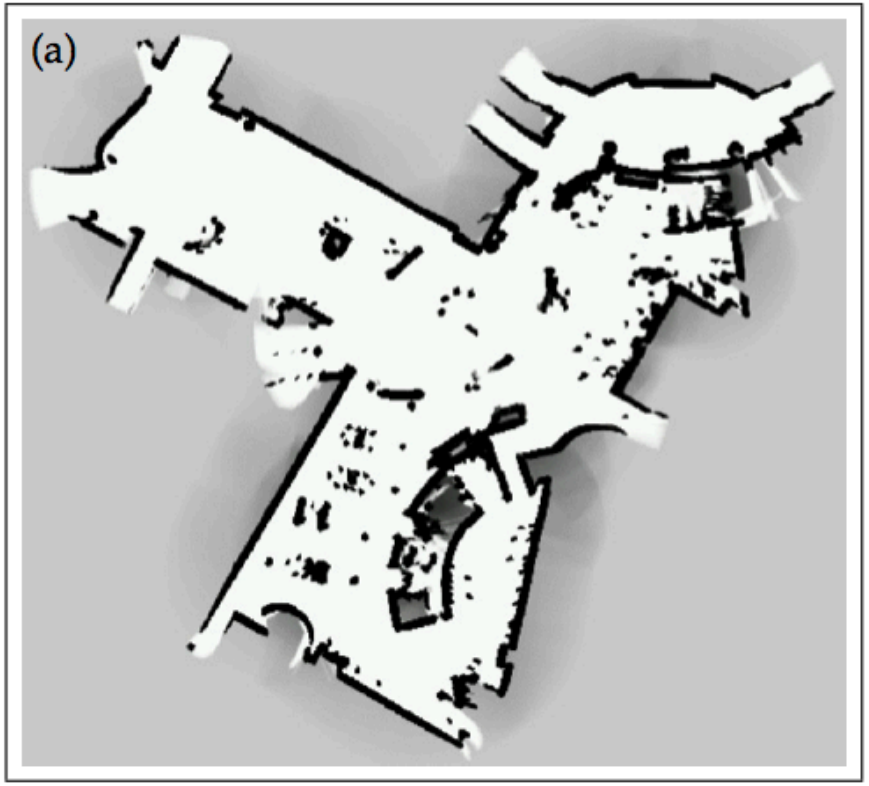
- Individual measurements are independent given the robot position

$$p(z_t \mid x_t, m) = \prod_{i=1}^{k} p(z_t^i \mid x_t, m)$$

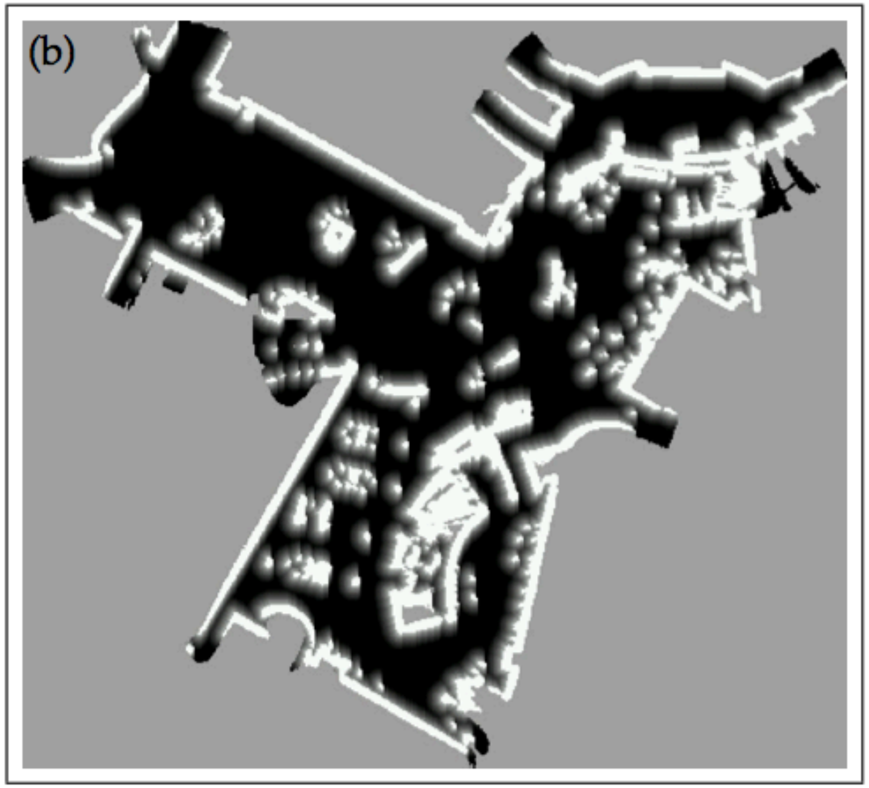
Beam-Endpoint Model



Beam-Endpoint Model



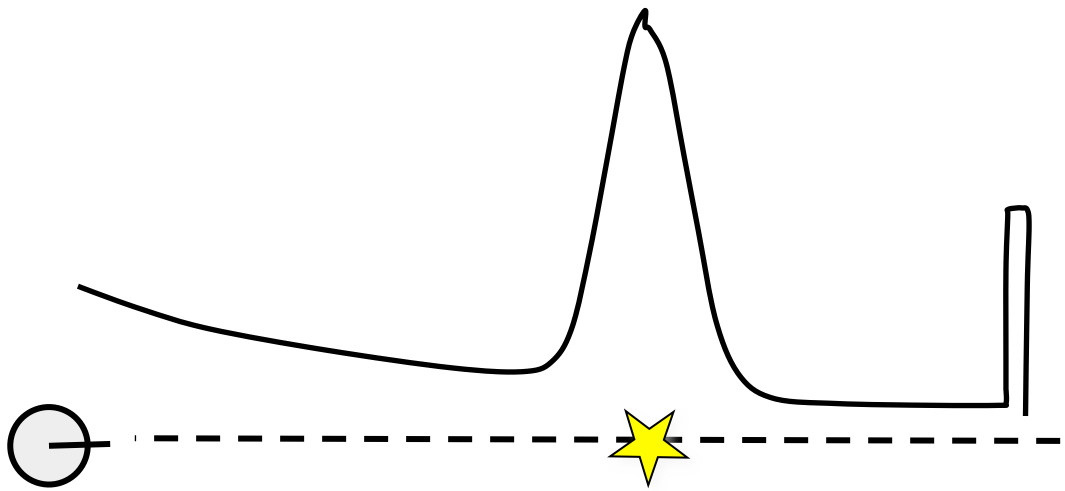
map



likelihood field

Ray-cast Model

- Ray-cast model considers the first obstacle long the line of sight
- Mixture of four models



Model for Perceiving Landmarks with Range-Bearing Sensors

- Range-bearing $z_t^i = (r_t^i, \phi_t^i)^T$
- Robot's pose $(x, y, \theta)^T$
- Observation of feature j at location $(m_{j,x}, m_{j,y})^T$

$$\begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix} + Q_t$$

Summary

- Bayes filter is a framework for state estimation
- Motion and sensor model are the central models in the Bayes filter
- Standard models for robot motion and laser-based range sensing

Literature

On the Bayes filter

- Thrun et al. "Probabilistic Robotics", Chapter 2
- Course: Introduction to Mobile Robotics, Chapter 5

On motion and observation models

- Thrun et al. "Probabilistic Robotics", Chapters 5 & 6
- Course: Introduction to Mobile Robotics, Chapters 6 & 7