

# Robot Mapping

## A Short Introduction to Homogeneous Coordinates

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**AiS** Autonomous  
Intelligent  
Systems

# Motivation

- Cameras generate a projected image of the world
- Euclidian geometry is suboptimal to describe the central projection
- In Euclidian geometry, the math can get difficult
- Projective geometry is an alternative algebraic representation of geometric objects and transformations
- Math becomes simpler

# Projective Geometry

- Projective geometry does not change the geometric relations
- Computations can also be done in Euclidian geometry (but more difficult)

# Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent affine transformations and projective transformations

# Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
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- Points at infinity can be represented using finite coordinates
- **A single matrix can represent affine transformations and projective transformations**

# Homogeneous Coordinates

## Definition

- The representation  $\mathbf{x}$  of a geometric object is homogeneous if  $\mathbf{x}$  and  $\lambda\mathbf{x}$  represent the same object for  $\lambda \neq 0$

## Example

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# From Homogeneous to Euclidian Coordinates

homogeneous

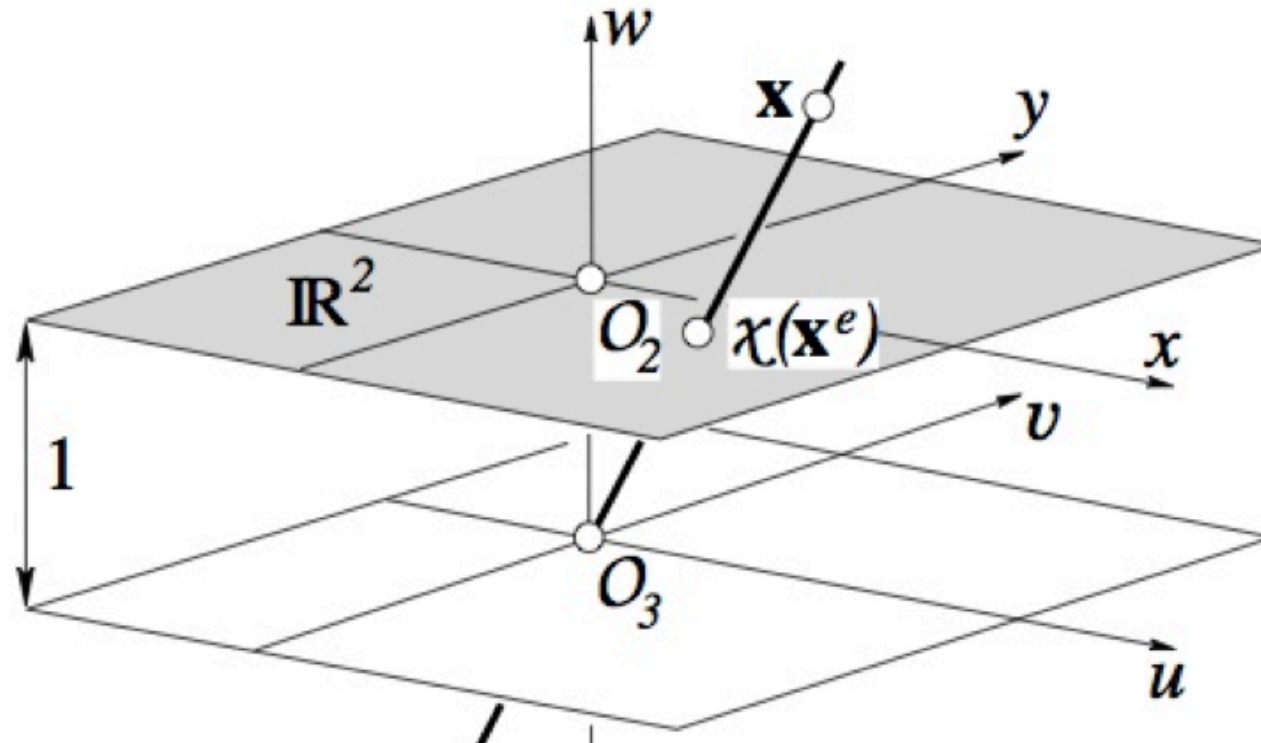
$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Euclidian

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

# From Homogeneous to Euclidian Coordinates



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



# Center of the Coordinate System

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{O}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

# Infinitively Distant Objects

- It is possible to explicitly model infinitively distant points with finite coordinates

$$\mathbf{x}_{\infty} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- Great tool when working with bearing-only sensors such as cameras

# 3D Points

- Analogous for 3D points

homogeneous

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} u/t \\ v/t \\ w/t \\ 1 \end{bmatrix}$$

Euclidian

$$\rightarrow \begin{bmatrix} u/t \\ v/t \\ w/t \end{bmatrix}$$

# Transformations

- A projective transformation is a invertible linear mapping

$$\mathbf{x}' = M\mathbf{x}$$

# Important Transformations ( $\mathbb{P}^3$ )

- General projective mapping

$$\mathbf{x}' = M \mathbf{x}$$

$4 \times 4$

- Translation: 3 parameters  
(3 translations)

$$M = \lambda \begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$

$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

# Important Transformations ( $\mathbb{P}^3$ )

- Rotation: 3 parameters  
(3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

rotation  
matrix

# Recap – Rotation Matrices

$$R^{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_x^{3D}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \quad R_y^{3D}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z^{3D}(\kappa) = \begin{bmatrix} \cos(\kappa) & -\sin(\kappa) & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{3D}(\omega, \phi, \kappa) = R_z^{3D}(\kappa)R_y^{3D}(\phi)R_x^{3D}(\omega)$$

# Important Transformations ( $\mathbb{P}^3$ )

- Rotation: 3 parameters  
(3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params  
(3 translation + 3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$



# Important Transformations ( $\mathbb{P}^3$ )







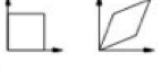
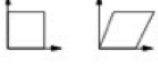


- Similarity transformation: 7 params  
(3 trans + 3 rot + 1 scale)

$$M = \lambda \begin{bmatrix} mR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Affine transformation: 12 parameters  
(3 trans + 3 rot + 3 scale + 3 shear)

$$M = \lambda \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

# Transformations in $\mathbb{P}^2$

2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & t \\ \mathbf{0}^T & 1 \end{bmatrix}$
Mirroring at $y$ -axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos \varphi & -\sin \varphi & t_x \\ \sin \varphi & \cos \varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & t \\ \mathbf{0}^T & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & t \\ \mathbf{0}^T & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & t \\ \mathbf{0}^T & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & t \\ p^T & 1/\lambda \end{bmatrix}$

[Courtesy by K. Schindler] 18

# Transformations

- Inverting a transformation

$$\mathbf{x}' = M\mathbf{x}$$

$$\mathbf{x} = M^{-1}\mathbf{x}'$$

- Chaining transformations via matrix products (not commutative)

$$\mathbf{x}' = M_1 M_2 \mathbf{x}$$

$$\neq M_2 M_1 \mathbf{x}$$

# Motions

- We will express motions (rotations and translations) using H.C.

$$M = \lambda \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{x}' &= M_1 M_2 \mathbf{x} \\ &\neq M_2 M_1 \mathbf{x} \end{aligned}$$

# Conclusion

- Homogeneous coordinates are an alternative representation for geometric objects
- Equivalence up to scale
$$\mathbf{x} \equiv \lambda \mathbf{x} \text{ with } \lambda \neq 0$$
- Modeled through an extra dimension
- Homogeneous coordinates can simplify mathematical expressions
- We often use it to represent the motion of objects

# Literature

## TOPIC

- Wikipedia as a good summary on homogeneous coordinates:  
[http://en.wikipedia.org/wiki/Homogeneous\\_coordinates](http://en.wikipedia.org/wiki/Homogeneous_coordinates)