Robot Mapping

A Short Introduction to Homogeneous Coordinates

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Motivation

- Cameras generate a projected image of the world
- Euclidian geometry is suboptimal to describe the central projection
- In Euclidian geometry, the math can get difficult
- Projective geometry is an alternative algebraic representation of geometric objects and transformations
- Math becomes simpler

Projective Geometry

- Projective geometry does not change the geometric relations
- Computations can also be done in Euclidian geometry (but more difficult)

Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent affine transformations and projective transformations

Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
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- Points at infinity can be represented using finite coordinates
- **A single matrix can represent affine transformations and projective transformations**

Homogeneous Coordinates

Definition

- The representation $\mathbf{x}$ of a geometric object is homogeneous if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

From Homogeneous to Euclidian Coordinates

homogeneous

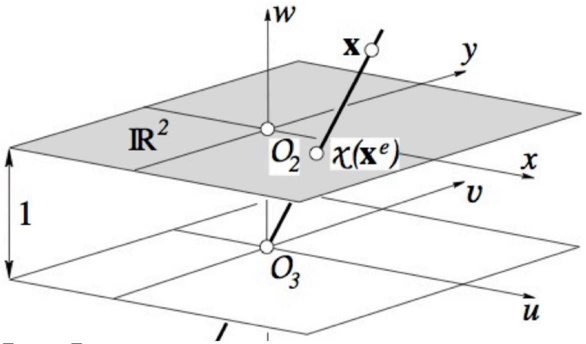
$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Euclidian

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From Homogeneous to Euclidian Coordinates



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

[Courtesy by K. Schindler]

Center of the Coordinate System

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{O}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Infinitively Distant Objects

- It is possible to explicitly model infinitively distant points with finite coordinates

$$\mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- Great tool when working with bearing-only sensors such as cameras

3D Points

- Analogous for 3D points

homogeneous — Euclidian

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} u/t \\ v/t \\ w/t \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/t \\ v/t \\ w/t \end{bmatrix}$$

Transformations

- A projective transformation is a invertible linear mapping

$$\mathbf{x}' = M\mathbf{x}$$

Important Transformations ($\mathbb{P}^3$)

- General projective mapping

$$\mathbf{x}' = \underset{4\times 4}{M}\ \mathbf{x}$$

- Translation: 3 parameters
  (3 translations)

$$M = \lambda \begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Important Transformations ($\mathbb{P}^3$)

- Rotation: 3 parameters
  (3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

rotation matrix

Recap – Rotation Matrices

$$R^{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_x^{3D}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \quad R_y^{3D}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z^{3D}(\kappa) = \begin{bmatrix} \cos(\kappa) & -\sin(\kappa) & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{3D}(\omega, \phi, \kappa) = R_z^{3D}(\kappa) R_y^{3D}(\phi) R_x^{3D}(\omega)$$

Important Transformations ($\mathbb{P}^3$)

- Rotation: 3 parameters
  (3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params
  (3 translation + 3 rotation)

$$M = \lambda \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Important Transformations ($\mathbb{P}^3$)

- Similarity transformation: 7 params
  (3 trans + 3 rot + 1 scale)

$$M = \lambda \begin{bmatrix} mR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Affine transformation: 12 parameters
  (3 trans + 3 rot + 3 scale + 3 sheer)

$$M = \lambda \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Transformations in $\mathbb{P}^2$

2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\mathsf{T} & 1/\lambda \end{bmatrix}$

[Courtesy by K. Schindler]

Transformations

- Inverting a transformation

$$\begin{aligned} \mathbf{x}' &= M\mathbf{x} \\ \mathbf{x} &= M^{-1}\mathbf{x}' \end{aligned}$$

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{x}' &= M_1 M_2 \mathbf{x} \\ &\neq M_2 M_1 \mathbf{x} \end{aligned}$$

Motions

- We will express motions (rotations and translations) using H.C.

$$M = \lambda \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{x}' &= M_1 M_2 \mathbf{x} \\ &\neq M_2 M_1 \mathbf{x} \end{aligned}$$

Conclusion

- Homogeneous coordinates are an alternative representation for geometric objects
- Equivalence up to scale

$$\mathbf{x} \equiv \lambda \mathbf{x} \text{ with } \lambda \neq 0$$

- Modeled through an extra dimension
- Homogeneous coordinates can simplify mathematical expressions
- We often use it to represent the motion of objects

Literature

TOPIC

- Wikipedia as a good summary on homogeneous coordinates:
  http://en.wikipedia.org/wiki/Homogeneous_coordinates