Robot Mapping

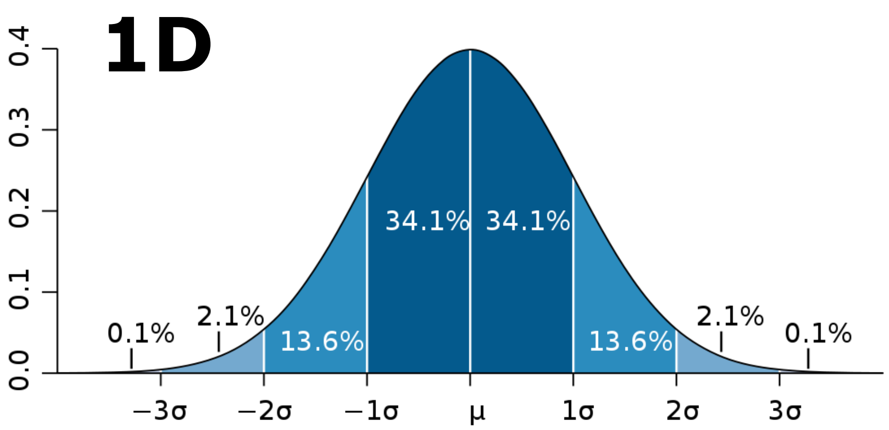
Extended Information Filter

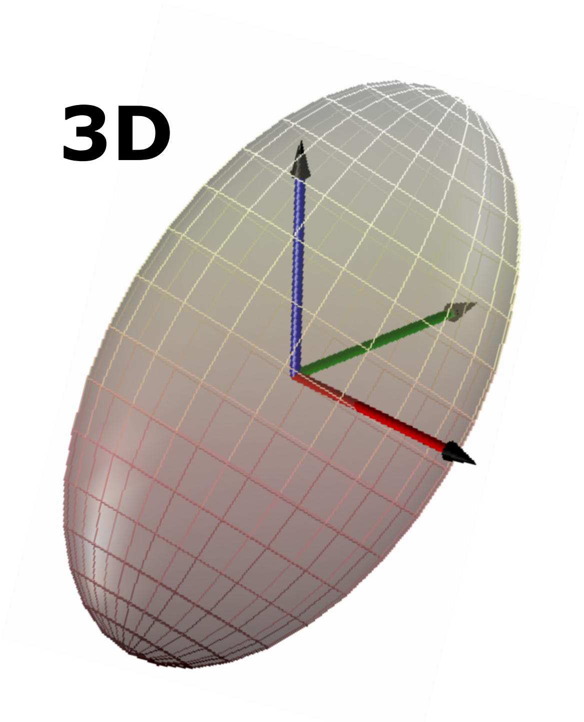
Cyrill Stachniss

Gaussians

- Gaussian described by moments μ, Σ

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)$$





Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** Ω and **information vector** ξ

Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** Ω

$$\Omega = \Sigma^{-1}$$

- and **information vector** ξ

$$\xi = \Sigma^{-1}\mu$$

Complete Parameterizations

moments

$$\Sigma = \Omega^{-1}$$

$$\mu = \Omega^{-1}\xi$$

canonical

$$\Omega = \Sigma^{-1}$$

$$\xi = \Sigma^{-1}\mu$$

Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&\qquad \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\right)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&\qquad \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\right) \\
&= \eta \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\right)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
& \qquad\qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \\
= \quad & \eta \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \\
= \quad & \eta \exp\Big(-\frac{1}{2}x^T\Omega x + x^T\xi\Big)
\end{aligned}
$$

Dual Representation

$$p(x) = \frac{\exp(-\frac{1}{2}\mu^T\xi)}{\det(2\pi\Omega^{-1})^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}x^T\Omega x + x^T\xi \Big)$$

canonical parameterization

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) \Big)$$

moments parameterization

From the Kalman Filter to the Information Filter

- Two parameterization for Gaussian
- We learned about Gaussian filtering with the Kalman filter in Chapter 3
- Kalman filtering in information from is called information filtering

Kalman Filter Algorithm

$$\textbf{Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$$
$$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$$
$$\mathit{return} \ \mu_t, \Sigma_t$$

Prediction Step (1)

- Transform $\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$
- Using $\Sigma_{t-1} = \Omega_{t-1}^{-1}$
- Leads to

$$\bar{\Omega}_t \quad = \quad (A_t \, \Omega_{t-1}^{-1} \, A_t^T + R_t)^{-1}$$

Prediction Step (2)

- Transform $\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$
- Using $\bar{\mu}_{t-1} = \Omega_{t-1}^{-1} \xi_{t-1}$
- Leads to

$$
\begin{aligned}
\bar{\xi}_t &= \bar{\Omega}_t (A_t \, \mu_{t-1} + B_t \, u_t) \\
&= \bar{\Omega}_t (A_t \, \Omega_{t-1}^{-1} \xi_{t-1} + B_t \, u_t)
\end{aligned}
$$

Information Filter Algorithm

1: **Information_filter**($\xi_{t-1}, \Omega_{t-1}, u_t, z_t$)**:**

2: $\bar{\Omega}_t = (A_t\, \Omega_{t-1}^{-1}\, A_t^T + R_t)^{-1}$

3: $\bar{\xi}_t = \bar{\Omega}_t (A_t\, \Omega_{t-1}^{-1}\, \xi_{t-1} + B_t\, u_t)$

4:

5:

6:

Correction Step

- Use the Bayes filter measurement update and replace the components

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

$$= \quad \eta\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t)\right)\ \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right)$$

Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
bel(x_t) &= \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t) \\
&= \eta\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t)\right)\ \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right) \\
&= \eta\ \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t) - \frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right)
\end{aligned}
$$

Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
bel(x_t) &= \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t) \\
&= \eta \exp\left(-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)\right) \exp\left(-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (x_t - \bar{\mu}_t)\right) \\
&= \eta \exp\left(-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t) - \frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (x_t - \bar{\mu}_t)\right) \\
&= \eta \exp\left(-\frac{1}{2} x_t^T C_t^T Q_t^{-1} C_t\, x_t + x_t^T C_t^T Q_t^{-1} z_t - \frac{1}{2} x_t^T \bar{\Omega}_t x_t + x_t^T \bar{\xi}_t\right)
\end{aligned}
$$

Correction Step

- Use the Bayes filter measurement update and replace the components

$$
\begin{aligned}
bel(x_t) &= \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t) \\
&= \eta \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t)\right) \exp\left(-\frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right) \\
&= \eta \exp\left(-\frac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t) - \frac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t)\right) \\
&= \eta \exp\left(-\frac{1}{2}\, x_t^T\, C_t^T\, Q_t^{-1}\, C_t\, x_t + x_t^T\, C_t^T\, Q_t^{-1}\, z_t - \frac{1}{2}\, x_t^T\, \bar{\Omega}_t x_t + x_t^T \bar{\xi}_t\right) \\
&= \eta \exp\left(-\frac{1}{2}\, x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t]}_{\Omega_t}\, x_t + x_t^T\, \underbrace{[C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t]}_{\xi_t}\right)
\end{aligned}
$$

Correction Step

- This results in an simple update rule

$$bel(x_t) = \eta \exp\left(-\frac{1}{2} x_t^T \underbrace{[C_t^T \, Q_t^{-1} \, C_t + \bar{\Omega}_t]}_{\Omega_t} x_t + x_t^T \underbrace{[C_t^T \, Q_t^{-1} \, z_t + \bar{\xi}_t]}_{\xi_t} \right)$$

$$\Downarrow$$

$$\begin{aligned} \Omega_t &= C_t^T \, Q_t^{-1} \, C_t + \bar{\Omega}_t \\ \xi_t &= C_t^T \, Q_t^{-1} \, z_t + \bar{\xi}_t \end{aligned}$$

Information Filter Algorithm

1: $\textbf{Information\_filter}(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)$**:**

2: $\bar{\Omega}_t = (A_t\ \Omega_{t-1}^{-1}\ A_t^T + R_t)^{-1}$

3: $\bar{\xi}_t = \bar{\Omega}_t(A_t\ \Omega_{t-1}^{-1}\ \xi_{t-1} + B_t\ u_t)$

4: $\Omega_t = C_t^T\ Q_t^{-1}\ C_t + \bar{\Omega}_t$

5: $\xi_t = C_t^T\ Q_t^{-1}\ z_t + \bar{\xi}_t$

6: $return\ \xi_t, \Omega_t$

Prediction and Correction

- Prediction

$$\begin{aligned}\bar{\Omega}_t &= (A_t\, \Omega_{t-1}^{-1}\, A_t^T + R_t)^{-1} \\ \bar{\xi}_t &= \bar{\Omega}_t(A_t\, \Omega_{t-1}^{-1}\, \xi_{t-1} + B_t\, u_t)\end{aligned}$$

- Correction

$$\begin{aligned}\Omega_t &= C_t^T\, Q_t^{-1}\, C_t + \bar{\Omega}_t \\ \xi_t &= C_t^T\, Q_t^{-1}\, z_t + \bar{\xi}_t\end{aligned}$$

Discuss differences to the KF!

Complexity

- Kalman filter
  - Efficient prediction step: $\mathcal{O}(n^2)^*$
  - Costly correction step: $\mathcal{O}(n^2 + k^{2.4})$
- Information filter
  - Costly prediction step: $\mathcal{O}(n^{2.4})$
  - Efficient correction step: $\mathcal{O}(n^2)^*$
- Transformation between both parameterizations is costly: $\mathcal{O}(n^{2.4})$

*Potentially faster, especially for SLAM; depending on type of control and observation

Extended Information Filter

- As the Kalman filter, the information filter suffers from the linear models
- The extended information filter (EIF) uses a similar trick as the EKF
- Linearization of the motion and observation function

Linearization of the EIF

- Taylor approximation analog to the EKF (see Chapter 3)

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + G_t \,(x_{t-1} - \mu_{t-1}) \\ h(x_t) &\approx h(\bar{\mu}_t) + H_t \,(x_t - \bar{\mu}_t) \end{aligned}$$

- with the Jacobians G_t and H_t

Prediction: From EKF of EIF

- Substitution of the moments brings us from the EKF

$$
\begin{aligned}
\bar{\Sigma}_t &= G_t \, \Sigma_{t-1} \, G_t^T + R_t \\
\bar{\mu}_t &= g(u_t, \mu_{t-1})
\end{aligned}
$$

- to the EIF

$$
\begin{aligned}
\bar{\Omega}_t &= (G_t \, \Omega_{t-1}^{-1} \, G_t^T + R_t)^{-1} \\
\bar{\xi}_t &= \bar{\Omega}_t \, g(u_t, \Omega_{t-1}^{-1} \, \xi_{t-1})
\end{aligned}
$$

Prediction: From EKF of EIF

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$

3: $\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$

1: **Extended_information_filter**$(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)$**:**

2: $\mu_{t-1} = \Omega_{t-1}^{-1}\, \xi_{t-1}$

3: $\bar{\Omega}_t = (G_t\, \Omega_{t-1}^{-1}\, G_t^T + R_t)^{-1}$

4: $\bar{\mu}_t = g(u_t, \mu_{t-1})$

5: $\bar{\xi}_t = \bar{\Omega}_t\, \bar{\mu}_t$

Correction Step of the EIF

- As from the KF to IF transition, use substitute the moments in the measurement update

$$
\begin{aligned}
bel(x_t) &= \eta \ \exp\left(-\frac{1}{2}\,(z_t - h(\bar{\mu}_t) - H_t\,(x_t - \bar{\mu}_t))^T\,Q_t^{-1}\right.\\
&\qquad \left.(z_t - h(\bar{\mu}_t) - H_t\,(x_t - \bar{\mu}_t)) - \frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right)
\end{aligned}
$$

- This leads to

$$
\begin{aligned}
\Omega_t &= \bar{\Omega}_t + H_t^T\,Q_t^{-1}\,H_t \\
\xi_t &= \bar{\xi}_t + H_t^T\,Q_t^{-1}\,(z_t - h(\bar{\mu}_t) + H_t\,\bar{\mu}_t)
\end{aligned}
$$

Extended Information Filter

$$\textbf{Extended\_information\_filter}(\xi_{t-1}, \Omega_{t-1}, u_t, z_t)\textbf{:}$$

$$\mu_{t-1} = \Omega_{t-1}^{-1}\ \xi_{t-1}$$

$$\bar{\Omega}_t = (G_t\ \Omega_{t-1}^{-1}\ G_t^T + R_t)^{-1}$$

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

$$\bar{\xi}_t = \bar{\Omega}_t\ \bar{\mu}_t$$

$$\Omega_t = \bar{\Omega}_t + H_t^T\ Q_t^{-1}\ H_t$$

$$\xi_t = \bar{\xi}_t + H_t^T\ Q_t^{-1}\ (z_t - h(\bar{\mu}_t) + H_t\ \bar{\mu}_t)$$

$$return\ \xi_t, \Omega_t$$

EIF vs. EKF

- The EIF is the EKF in information form
- Complexity of the prediction and corrections steps differs
- Same expressiveness than the EKF
- Unscented transform can also be used
- Reported to be numerically more stable than the EKF
- In practice, the EKF is more popular than the EIF

Summary

- Gaussians can also be represented using the canonical parameterization
- Allow for filtering in information form
- Information filter vs. Kalman filter
- KF: efficient prediction, slow correction
- IF: slow prediction, efficient correction
- The application determines which filter is superior!

Literature

Extended Information Filter

- Thrun et al.: "Probabilistic Robotics", Chapter 3.5