Robot Mapping

Unscented Kalman Filter

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KF, EKF and UKF

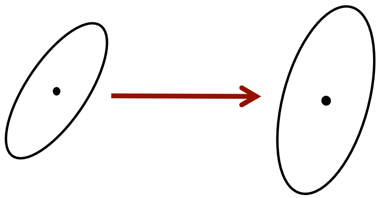
- Kalman filter requires linear models
- EKF linearizes via Taylor expansion

Is there a better way to linearize?

Unscented Transform

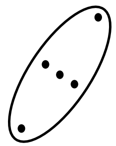
Unscented Kalman Filter (UKF)

Taylor Approximation (EKF)



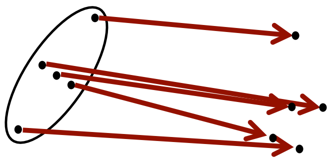
Linearization of the non-linear function through Taylor expansion

Unscented Transform



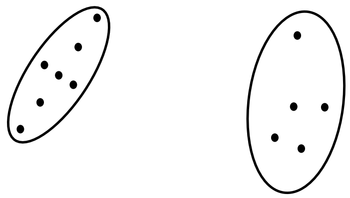
Compute a set of (so-called) sigma points

Unscented Transform



Transform each sigma point
through the non-linear function

Unscented Transform



Compute Gaussian from the
transformed and weighted points

Unscented Transform Overview

- Compute a set of sigma points
- Each sigma points has a weight
- Transform the point through the non-linear function
- Compute a Gaussian from weighted points

- **Avoids to linearize around the mean as the EKF does**

Sigma Points

- How to choose the sigma points?
- How to set the weights?

Sigma Points Properties

- How to choose the sigma points?
- How to set the weights?
- Select $\mathcal{X}^{[i]}, w^{[i]}$ so that:

$$\begin{aligned} \sum_i w^{[i]} &= 1 \\ \mu &= \sum_i w^{[i]} \mathcal{X}^{[i]} \\ \Sigma &= \sum_i w^{[i]} (\mathcal{X}^{[i]} - \mu)(\mathcal{X}^{[i]} - \mu)^T \end{aligned}$$

- There is no unique solution for $\mathcal{X}^{[i]}, w^{[i]}$

Sigma Points

- Choosing the sigma points

$$\mathcal{X}^{[0]} = \mu$$

First sigma point is the mean

Sigma Points

- Choosing the sigma points

$$\begin{aligned} \mathcal{X}^{[0]} &= \mu \\ \mathcal{X}^{[i]} &= \mu + \left(\sqrt{(n+\lambda)\,\Sigma}\right)_i \quad \text{for } i = 1, \ldots, n \\ \mathcal{X}^{[i]} &= \mu - \left(\sqrt{(n+\lambda)\,\Sigma}\right)_{i-n} \quad \text{for } i = n+1, \ldots, 2n \end{aligned}$$

matrix square root

column vectors

dimensionality

scaling parameter

Matrix Square Root

- Defined as S with $\Sigma = SS$
- Computed via diagonalization

$$\begin{aligned} \Sigma &= VDV^{-1} \\ &= V \begin{pmatrix} d_{11} & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & d_{nn} \end{pmatrix} V^{-1} \\ &= V \begin{pmatrix} \sqrt{d_{11}} & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & \sqrt{d_{nn}} \end{pmatrix}^2 V^{-1} \end{aligned}$$

Matrix Square Root

- Thus, we can define

$$S = V \underbrace{\begin{pmatrix} \sqrt{d_{11}} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sqrt{d_{nn}} \end{pmatrix}}_{D^{1/2}} V^{-1}$$

- so that

$$SS = (VD^{1/2}V^{-1})(VD^{1/2}V^{-1}) = VDV^{-1} = \Sigma$$

Cholesky Matrix Square Root

- Alternative definition of the matrix square root

$$L \text{ with } \Sigma = LL^T$$

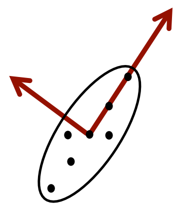
- Result of the Cholesky decomposition
- Numerically stable solution
- Often used in UKF implementations
- L and Σ have the same Eigenvectors

Sigma Points and Eigenvectors

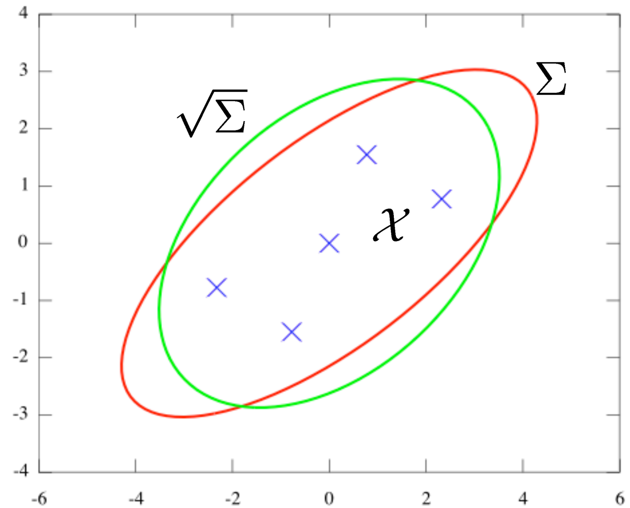
- Sigma point **can** but **do not have to** lie on the main axes of Σ

$$\mathcal{X}^{[i]} = \mu + \left(\sqrt{(n+\lambda)\,\Sigma}\right)_i \quad \text{for } i = 1, \dots, n$$

$$\mathcal{X}^{[i]} = \mu - \left(\sqrt{(n+\lambda)\,\Sigma}\right)_{i-n} \quad \text{for } i = n+1, \dots, 2n$$



Sigma Points Example



Sigma Point Weights

- Weight sigma points

for computing the mean

parameters

$$
\begin{aligned}
w_m^{[0]} &= \frac{\lambda}{n+\lambda} \\
w_c^{[0]} &= w_m^{[0]} + (1-\alpha^2+\beta) \\
w_m^{[i]} = w_c^{[i]} &= \frac{1}{2(n+\lambda)} \quad \text{for } i = 1, \ldots, 2n
\end{aligned}
$$

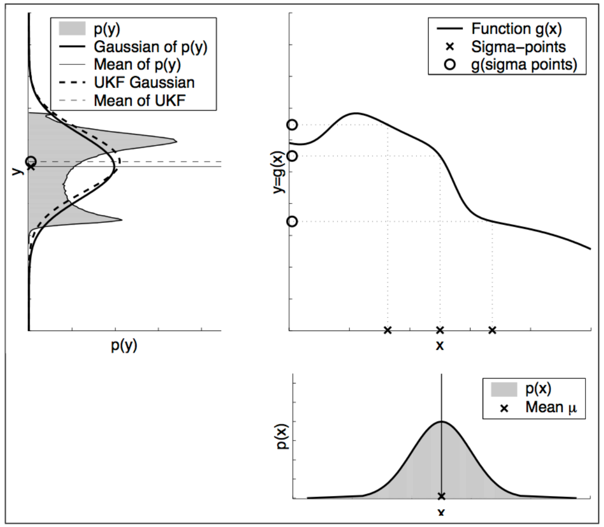
for computing the covariance

Recover the Gaussian

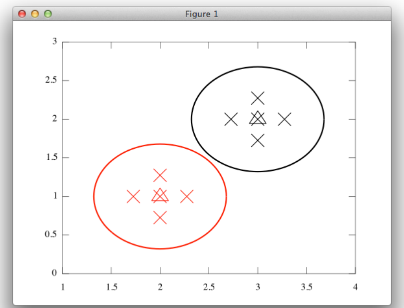
- Compute Gaussian from weighted and transformed points

$$
\begin{aligned}
\mu' &= \sum_{i=0}^{2n} w_m^{[i]} \, g(\mathcal{X}^{[i]}) \\
\Sigma' &= \sum_{i=0}^{2n} w_c^{[i]} \, (g(\mathcal{X}^{[i]}) - \mu')(g(\mathcal{X}^{[i]}) - \mu')^T
\end{aligned}
$$

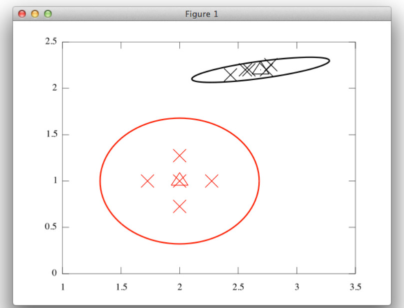
Example



Examples



$$g((x,y)^T) = \begin{pmatrix} x+1 \\ y+1 \end{pmatrix}^T$$



$$g((x,y)^T) = \begin{pmatrix} 1+x+\sin(2x)+\cos(y) \\ 2+0.2y \end{pmatrix}^T$$

Unscented Transform Summary

- Sigma points

$$
\begin{aligned}
\mathcal{X}^{[0]} &= \mu \\
\mathcal{X}^{[i]} &= \mu + \left(\sqrt{(n+\lambda)\,\Sigma}\right)_i \quad \text{for } i = 1, \ldots, n \\
\mathcal{X}^{[i]} &= \mu - \left(\sqrt{(n+\lambda)\,\Sigma}\right)_{i-n} \quad \text{for } i = n+1, \ldots, 2n
\end{aligned}
$$

- Weights

$$
\begin{aligned}
w_m^{[0]} &= \frac{\lambda}{n+\lambda} \\
w_c^{[0]} &= w_m^{[0]} + (1 - \alpha^2 + \beta) \\
w_m^{[i]} = w_c^{[i]} &= \frac{1}{2(n+\lambda)} \quad \text{for } i = 1, \ldots, 2n
\end{aligned}
$$

UT Parameters

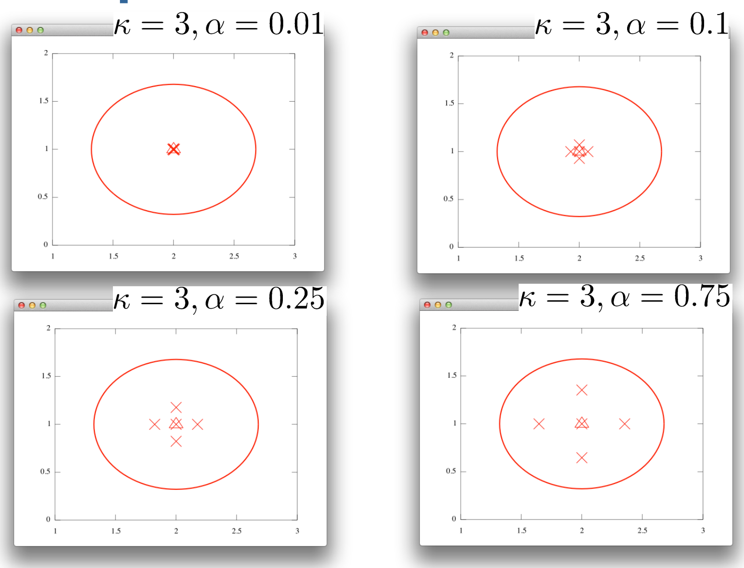
- Free parameters as there is no unique solution
- Scales Unscented Transform uses

$$
\begin{aligned}
\kappa &\geq 0 \\
\alpha &\in (0, 1] \\
\lambda &= \alpha^2 (n + \kappa) - n \\
\beta &= 2
\end{aligned}
$$

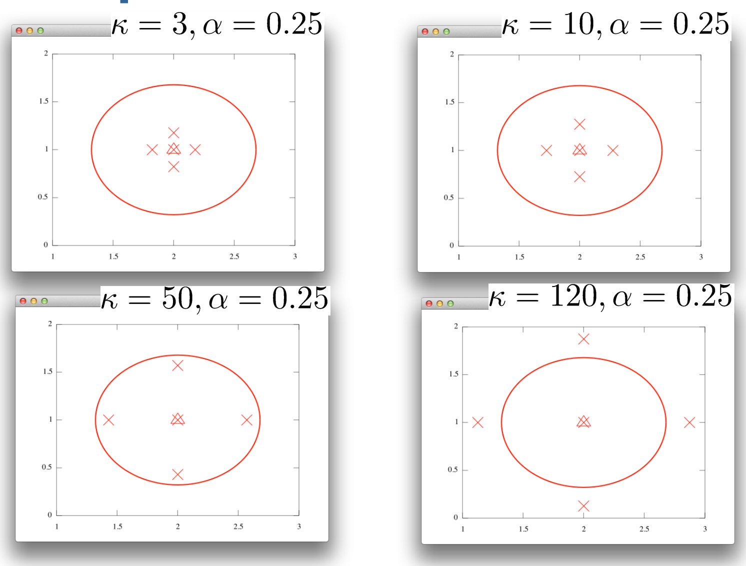
Influence how far the sigma points are away from the mean (κ, α)

Optimal choice for Gaussians ($\beta = 2$)

Examples



Examples



EKF Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$
3: $\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$
7: return μ_t, Σ_t

EKF to UKF – Prediction

Unscented
1: **~~Extended~~_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: $\bar{\mu}_t =$ replace this by sigma point
3: $\bar{\Sigma}_t =$ propagation of the motion

4: $K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$
7: return μ_t, Σ_t

UKF Algorithm – Prediction

1: **Unscented_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: $\mathcal{X}_{t-1} = (\mu_{t-1} \quad \mu_{t-1} + \gamma\sqrt{\Sigma_{t-1}} \quad \mu_{t-1} - \gamma\sqrt{\Sigma_{t-1}})$
3: $\bar{\mathcal{X}}_t^* = g(u_t, \mathcal{X}_{t-1})$
4: $\bar{\mu}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{X}}_t^{*[i]}$
5: $\bar{\Sigma}_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{*[i]} - \bar{\mu}_t)(\bar{\mathcal{X}}_t^{*[i]} - \bar{\mu}_t)^T + R_t$

EKF to UKF – Correction

Unscented
1: **~~Extended~~_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: $\bar{\mu}_t =$ replace this by sigma point
3: $\bar{\Sigma}_t =$ propagation of the motion

use sigma point propagation for the
expected observation and Kalman gain

5: $\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$
6: $\Sigma_t = \bar{\Sigma}_t - K_t\, S_t\, K_t^T$
7: return μ_t, Σ_t

UKF Algorithm – Correction (1)

$$
\begin{array}{ll}
6: & \bar{\mathcal{X}}_t = (\bar{\mu}_t \quad \bar{\mu}_t + \gamma\sqrt{\bar{\Sigma}_t} \quad \bar{\mu}_t - \gamma\sqrt{\bar{\Sigma}_t}) \\
7: & \bar{\mathcal{Z}}_t = h(\bar{\mathcal{X}}_t) \\
8: & \hat{z}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{Z}}_t^{[i]} \\
9: & S_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T + Q_t \\
10: & \bar{\Sigma}_t^{x,z} = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{[i]} - \bar{\mu}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T \\
11: & K_t = \bar{\Sigma}_t^{x,z} \; S_t^{-1}
\end{array}
$$

UKF Algorithm – Correction (1)

$$
\begin{array}{ll}
6: & \bar{\mathcal{X}}_t = (\bar{\mu}_t \quad \bar{\mu}_t + \gamma\sqrt{\bar{\Sigma}_t} \quad \bar{\mu}_t - \gamma\sqrt{\bar{\Sigma}_t}) \\
7: & \bar{\mathcal{Z}}_t = h(\bar{\mathcal{X}}_t) \\
8: & \hat{z}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{Z}}_t^{[i]} \\
9: & S_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T + Q_t \\
10: & \bar{\Sigma}_t^{x,z} = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{[i]} - \bar{\mu}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T \\
11: & K_t = \bar{\Sigma}_t^{x,z} \; S_t^{-1}
\end{array}
$$

$$K_t = \overbrace{\bar{\Sigma}_t \; H_t^T}^{\bar{\Sigma}_t^{x,z}} \overbrace{(H_t \; \bar{\Sigma}_t \; H_t^T + Q_t)}^{S_t}{}^{-1}$$

(from EKF)

UKF Algorithm – Correction (2)

$$
\begin{array}{ll}
6: & \bar{\mathcal{X}}_t = (\bar{\mu}_t \quad \bar{\mu}_t + \gamma\sqrt{\bar{\Sigma}_t} \quad \bar{\mu}_t - \gamma\sqrt{\bar{\Sigma}_t}) \\
7: & \bar{\mathcal{Z}}_t = h(\bar{\mathcal{X}}_t) \\
8: & \hat{z}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{Z}}_t^{[i]} \\
9: & S_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T + Q_t \\
10: & \bar{\Sigma}_t^{x,z} = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{[i]} - \bar{\mu}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T \\
11: & K_t = \bar{\Sigma}_t^{x,z} \; S_t^{-1} \\
12: & \mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t) \\
13: & \Sigma_t = \bar{\Sigma}_t - K_t \; S_t \; K_t^T \\
14: & \text{return } \mu_t, \Sigma_t
\end{array}
$$

UKF Algorithm – Correction (2)

$$
\begin{array}{ll}
6: & \bar{\mathcal{X}}_t = (\bar{\mu}_t \quad \bar{\mu}_t + \gamma\sqrt{\bar{\Sigma}_t} \quad \bar{\mu}_t - \gamma\sqrt{\bar{\Sigma}_t}) \\
7: & \bar{\mathcal{Z}}_t = h(\bar{\mathcal{X}}_t) \\
8: & \hat{z}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{Z}}_t^{[i]} \\
9: & S_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T + Q_t \\
10: & \bar{\Sigma}_t^{x,z} = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{[i]} - \bar{\mu}_t)(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t)^T \\
11: & K_t = \bar{\Sigma}_t^{x,z} \; S_t^{-1} \\
12: & \mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t) \\
13: & \Sigma_t = \bar{\Sigma}_t - K_t \; S_t \; K_t^T \\
14: & \text{return } \mu_t, \Sigma_t
\end{array}
$$

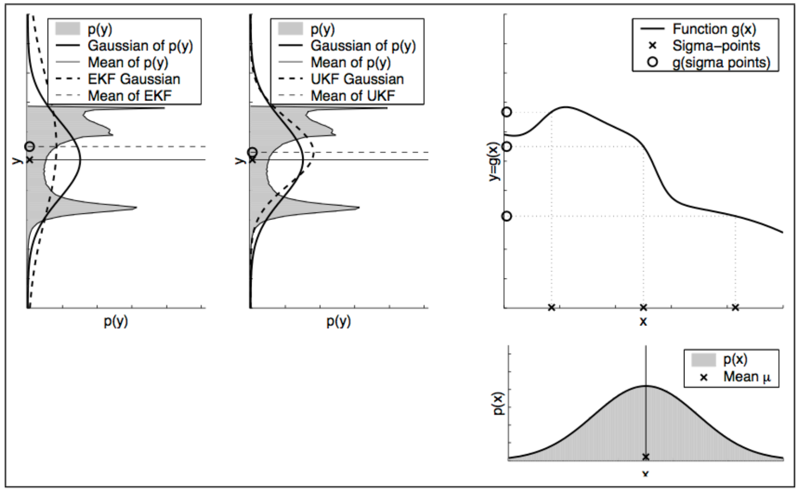
$$
\begin{aligned}
\Sigma_t &= (I - K_t H_t)\bar{\Sigma}_t \\
&= \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t \\
&= \bar{\Sigma}_t - K_t \left(\Sigma^{x,z}\right)^T \\
&= \bar{\Sigma}_t - K_t \left(\Sigma^{x,z} s_t^{-1} S_t\right)^T \\
&= \bar{\Sigma}_t - K_t \left(K_t S_t\right)^T \\
&= \bar{\Sigma}_t - K_t S_t^T K_t^T \\
&= \bar{\Sigma}_t - K_t S_t K_t^T
\end{aligned}
$$

(see next slide)

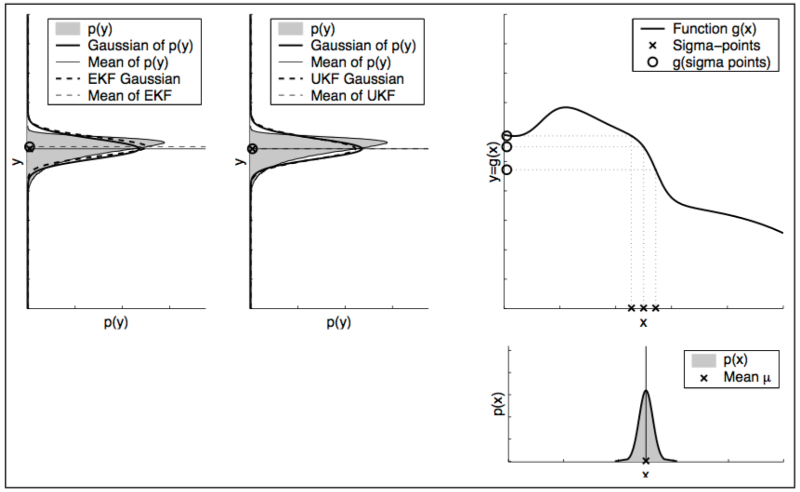
From EKF to UKF – Computing the Covariance

$$
\begin{aligned}
\Sigma_t &= (I - K_t H_t)\bar{\Sigma}_t \\
&= \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t \\
&= \bar{\Sigma}_t - K_t \left(\bar{\Sigma}^{x,z}\right)^T \\
&= \bar{\Sigma}_t - K_t \left(\bar{\Sigma}^{x,z} S_t^{-1} S_t\right)^T \\
&= \bar{\Sigma}_t - K_t \left(K_t S_t\right)^T \\
&= \bar{\Sigma}_t - K_t S_t^T K_t^T \\
&= \bar{\Sigma}_t - K_t S_t K_t^T
\end{aligned}
$$

UKF vs. EKF

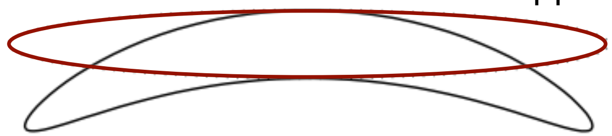


UKF vs. EKF (Small Covariance)

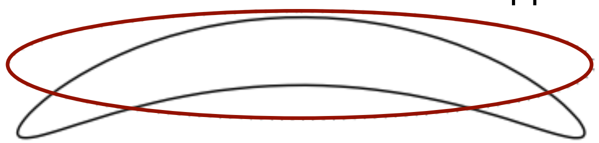


UKF vs. EKF – Banana Shape

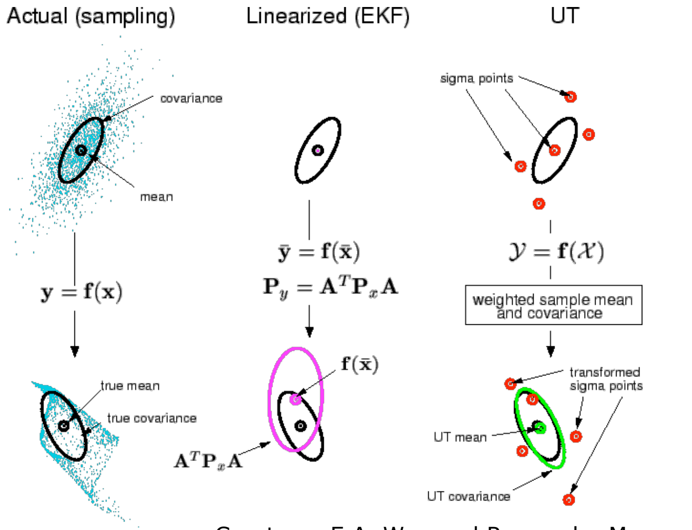
EKF approximation



UKF approximation



UKF vs. EKF



Courtesy: E.A. Wan and R. van der Merwe

UT/UKF Summary

- Unscented transforms as an alternative to linearization
- UT is a better approximation than Taylor expansion
- UT uses sigma point propagation
- Free parameters in UT
- UKF uses the UT in the prediction and correction step

UKF vs. EKF

- Same results as EKF for linear models
- Better approximation than EKF for non-linear models
- Differences often “somewhat small”
- No Jacobians needed for the UKF
- Same complexity class
- Slightly slower than the EKF
- Still requires Gaussian distributions

Literature

Unscented Transform and UKF

- Thrun et al.: “Probabilistic Robotics”, Chapter 3.4
- “A New Extension of the Kalman Filter to Nonlinear Systems” by Julier and Uhlmann, 1995
- “Dynamische Zustandsschätzung” by Fränken, 2006, pages 31-34