POMDPs: Partially Observable Markov Decision Processes

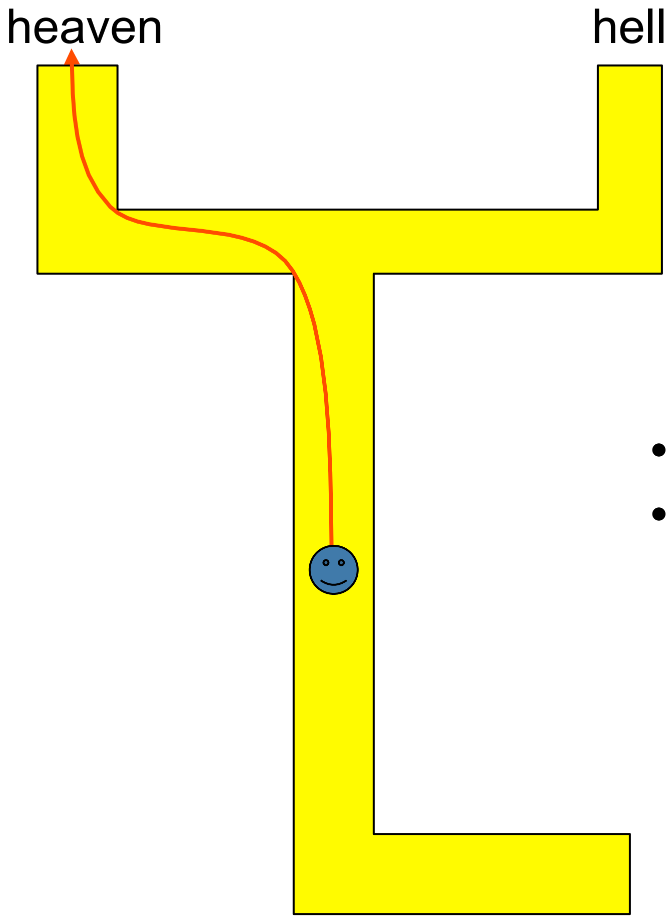
Robotics II

Wolfram Burgard

Types of Planning Problems

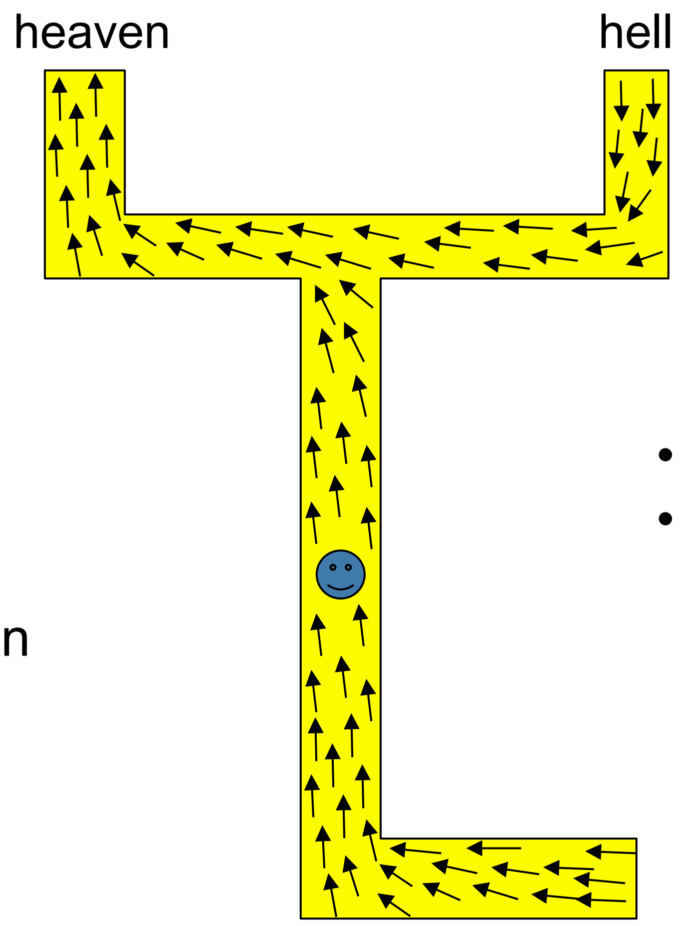
	State	Action Model
Classical Planning	observable	Deterministic, accurate
MDP, universal plans	observable	stochastic
POMDPs	partially observable	stochastic

Classical Planning



- World deterministic
- State observable

MDP-Style Planning

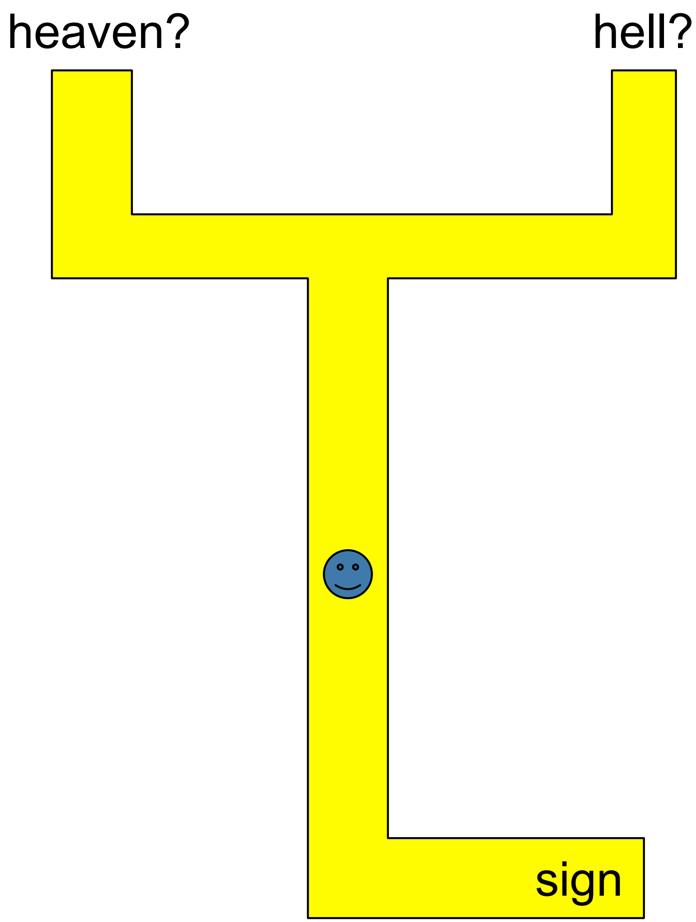


- Policy
- Universal Plan
- Navigation function

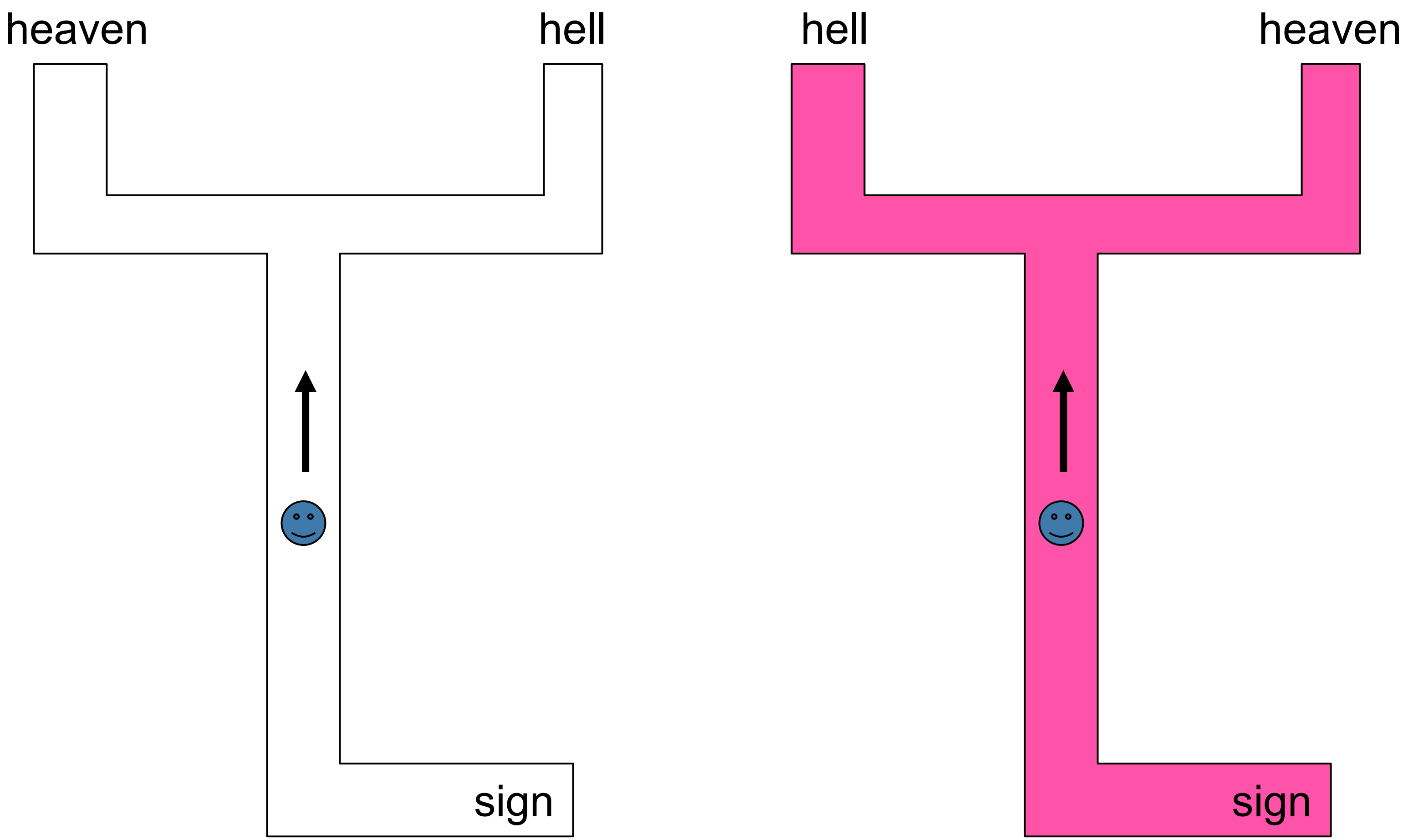
- World stochastic
- State observable

[Koditschek 87, Barto et al. 89]

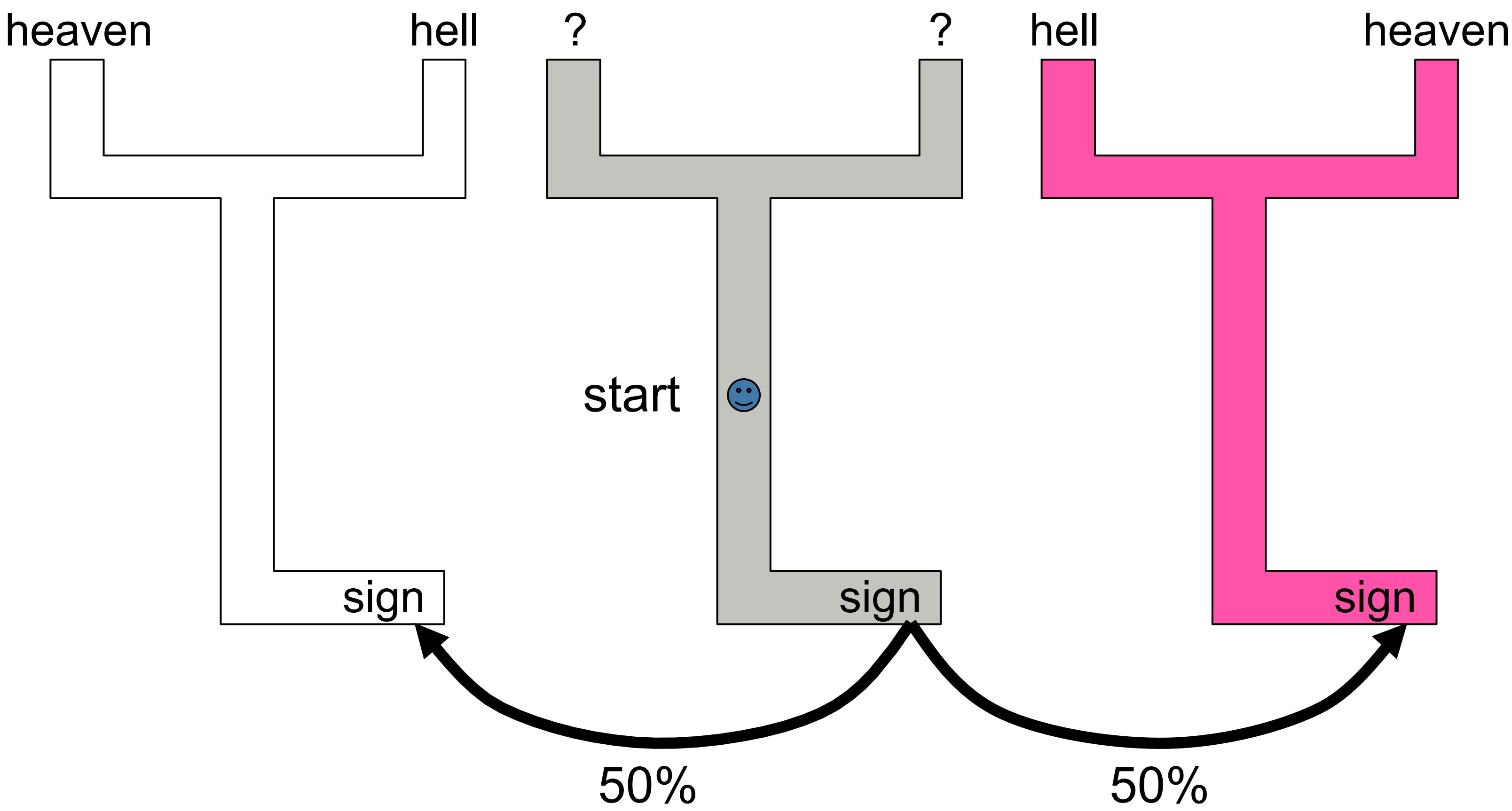
Stochastic, Partially Observable



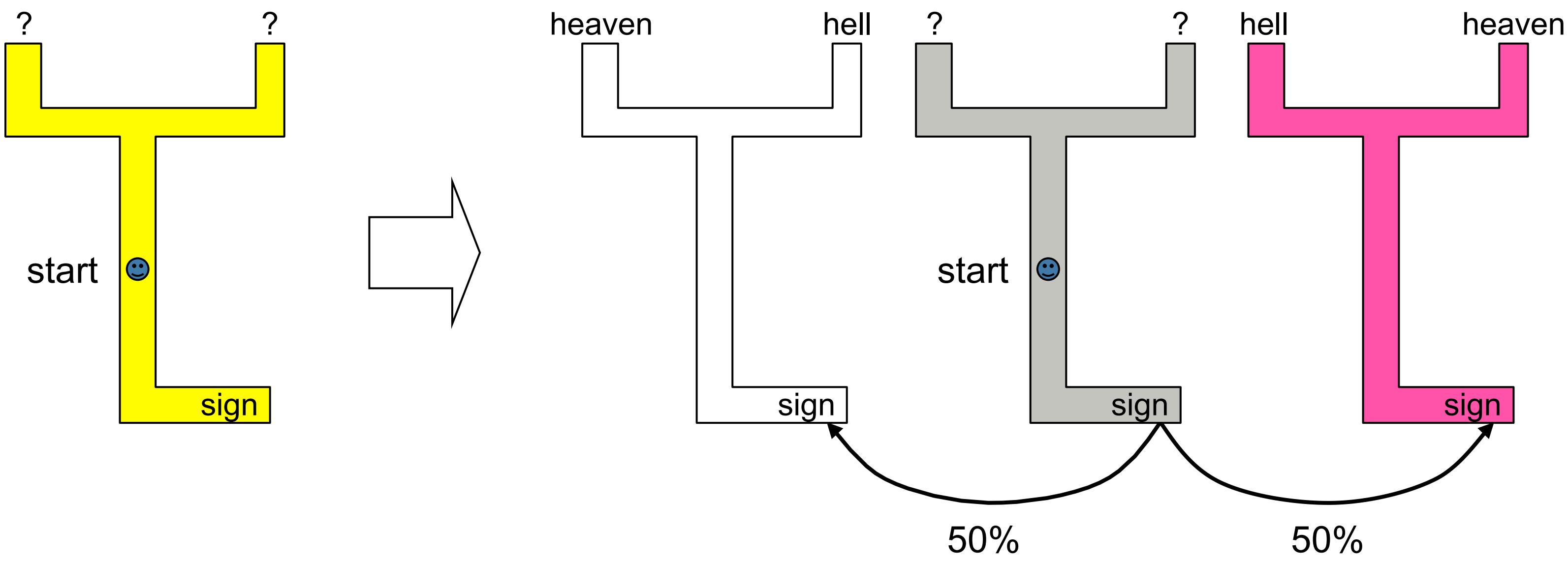
Stochastic, Partially Observable



Stochastic, Partially Observable



Stochastic, Partially Observable



Combination of Markov Decision Processes and Bayes Filtering

- Markov Decision Processes provide us with the optimal action given the state is known
- Recursive Bayes filtering provide us with an estimate about the current state of the system given all observations and actions carried out thus far.
- Can we extend MDPs to partially observable states using Recursive Bayes filtering?

Value Iteration

- Given this notation the value iteration formula is

$$V_T(x) \;=\; \gamma \max_u \left[r(x,u) + \int V_{T-1}(x')\, p(x' \mid u,x)\, dx' \right]$$

$$V_T(x) \;=\; \gamma \max_u \left[r(x,u) + \sum_{x'} V_{T-1}(x')\, p(x' \mid u,x) \right]$$

- with

$$V_1(b) \;=\; \gamma \max_u r(x,u)$$

POMDPs

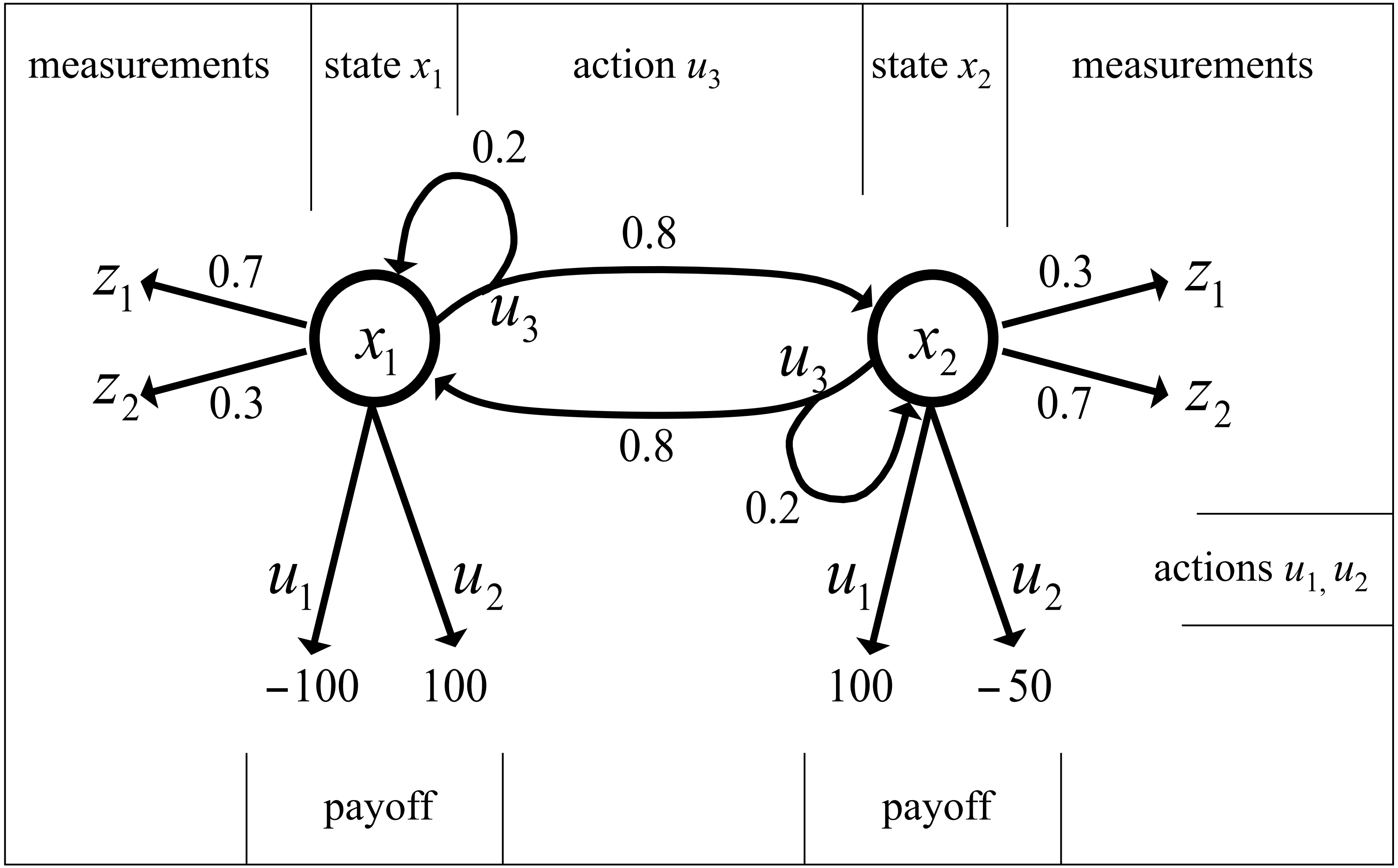
- In POMDPs we apply the very same idea as in MDPs.
- **Since the state is not observable**, the agent has to **make its decisions based on** the belief state which is a **posterior distribution over states**.
- Let b be the belief of the agent about the state under consideration.
- POMDPs compute a **value function over belief space**:

$$V_T(b) \;=\; \gamma \max_u \left[r(b,u) + \int V_{T-1}(b') p(b' \mid u, b) \, db' \right]$$

Problems

- Each belief is a probability distribution, thus, each value in a **POMDP is a function of an entire probability distribution**.
- **This is problematic, since probability distributions are continuous**.
- Additionally, we have to deal with the **huge complexity of belief spaces**.
- For **finite worlds** with finite state, action, and measurement spaces and finite horizons, however, we can **effectively represent the value functions by piecewise linear functions**.

An Illustrative Example



The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sense action that potentially leads to a state transition.

$$
\begin{aligned}
r(x_1, u_1) &= -100 & \qquad r(x_2, u_1) &= +100 \\
r(x_1, u_2) &= +100 & \qquad r(x_2, u_2) &= -50 \\
r(x_1, u_3) &= -1 & \qquad r(x_2, u_3) &= -1
\end{aligned}
$$

$$
\begin{aligned}
p(x_1'|x_1, u_3) &= 0.2 & \qquad p(x_2'|x_1, u_3) &= 0.8 \\
p(x_1'|x_2, u_3) &= 0.8 & \qquad p(z_2'|x_2, u_3) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(z_1|x_1) &= 0.7 & \qquad p(z_2|x_1) &= 0.3 \\
p(z_1|x_2) &= 0.3 & \qquad p(z_2|x_2) &= 0.7
\end{aligned}
$$

Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$
\begin{aligned}
r(b,u) &= E_x[r(x,u)] \\
&= \int r(x,u)p(x)\,dx \\
&= p_1\,r(x_1,u) + p_2\,r(x_2,u)
\end{aligned}
$$

Payoffs in Our Example (1)

- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by the corresponding probabilities

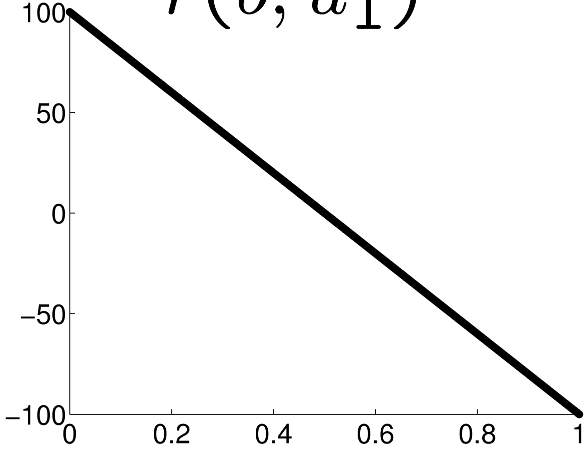
$$
\begin{aligned}
r(b, u_1) &= -100\, p_1 + 100\, p_2 \\
&= -100\, p_1 + 100\, (1 - p_1)
\end{aligned}
$$

$$r(b, u_2) = 100\, p_1 - 50\, (1 - p_1)$$

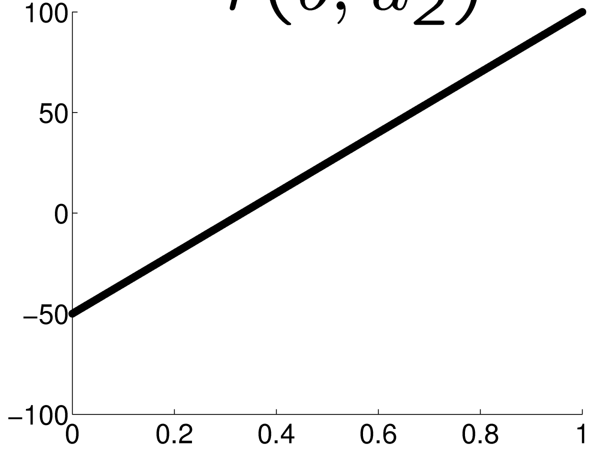
$$r(b, u_3) = -1$$

Payoffs in Our Example (2)

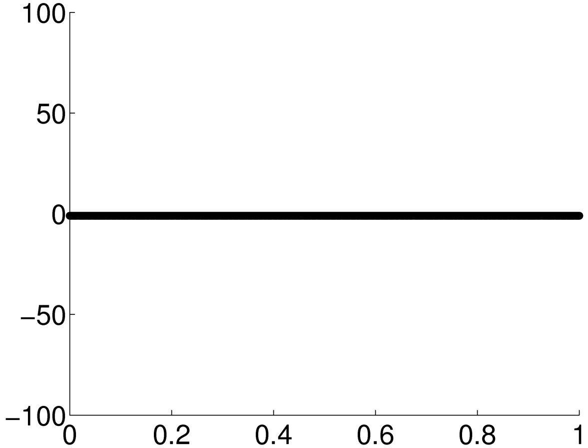
$$r(b, u_1)$$



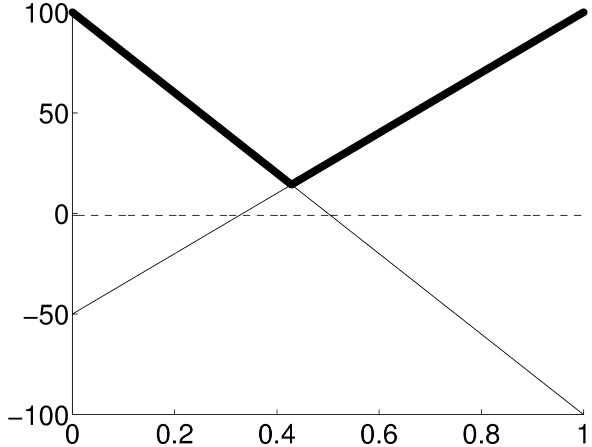
$$r(b, u_2)$$



$$r(b, u_3)$$



$$V_1(b) = max_u r(b, u)$$



The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \leq \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions in at each point

$$
\begin{aligned}
V_1(b) &= \max_u \, r(b,u) \\
&= \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\}
\end{aligned}
$$

- It is piecewise linear and convex.

Pruning

- If we carefully consider $V_1(b)$, we see that only the first to components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

$$V_1(b) = \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

Increasing the Time Horizon

- If we go over to a time horizon of T=2, the agent can also consider the sensing action u_3.
- Suppose we perceive z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$
\begin{aligned}
V_1(b \mid z_1) &= \max \left\{ \begin{array}{ll} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\} \\
&= \frac{1}{p(z_1)} \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$
\begin{aligned}
\bar{V}_1(b) &= E_z[V_1(b \mid z)] \\
&= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\
&= \max \left\{ \begin{array}{ll} -70\, p_1 & +30\, (1 - p_1) \\ 70\, p_1 & -15\, (1 - p_1) \end{array} \right\} \\
&\quad + \max \left\{ \begin{array}{ll} -30\, p_1 & +70\, (1 - p_1) \\ 30\, p_1 & -35\, (1 - p_1) \end{array} \right\}
\end{aligned}
$$

Resulting Value Function

- The four possible combinations yield the following function which then can be simplified and pruned.

$$\begin{aligned}
\bar{V}_1(b) &= \max \left\{ \begin{array}{llll} -70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ -70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \end{array} \right\} \\
&= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ +40\, p_1 & +55\,(1-p_1) \\ +100\, p_1 & -50\,(1-p_1) \end{array} \right\}
\end{aligned}$$

State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$
\begin{aligned}
p'_1 &= E_x[p(x_1 \mid x, u_3)] \\
&= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\
&= 0.2 p_1 + 0.8(1 - p_1) \\
&= 0.8 - 0.6 p_1
\end{aligned}
$$

Resulting Value Function after executing u_3

- Taking also the state transitions into account, we finally obtain.

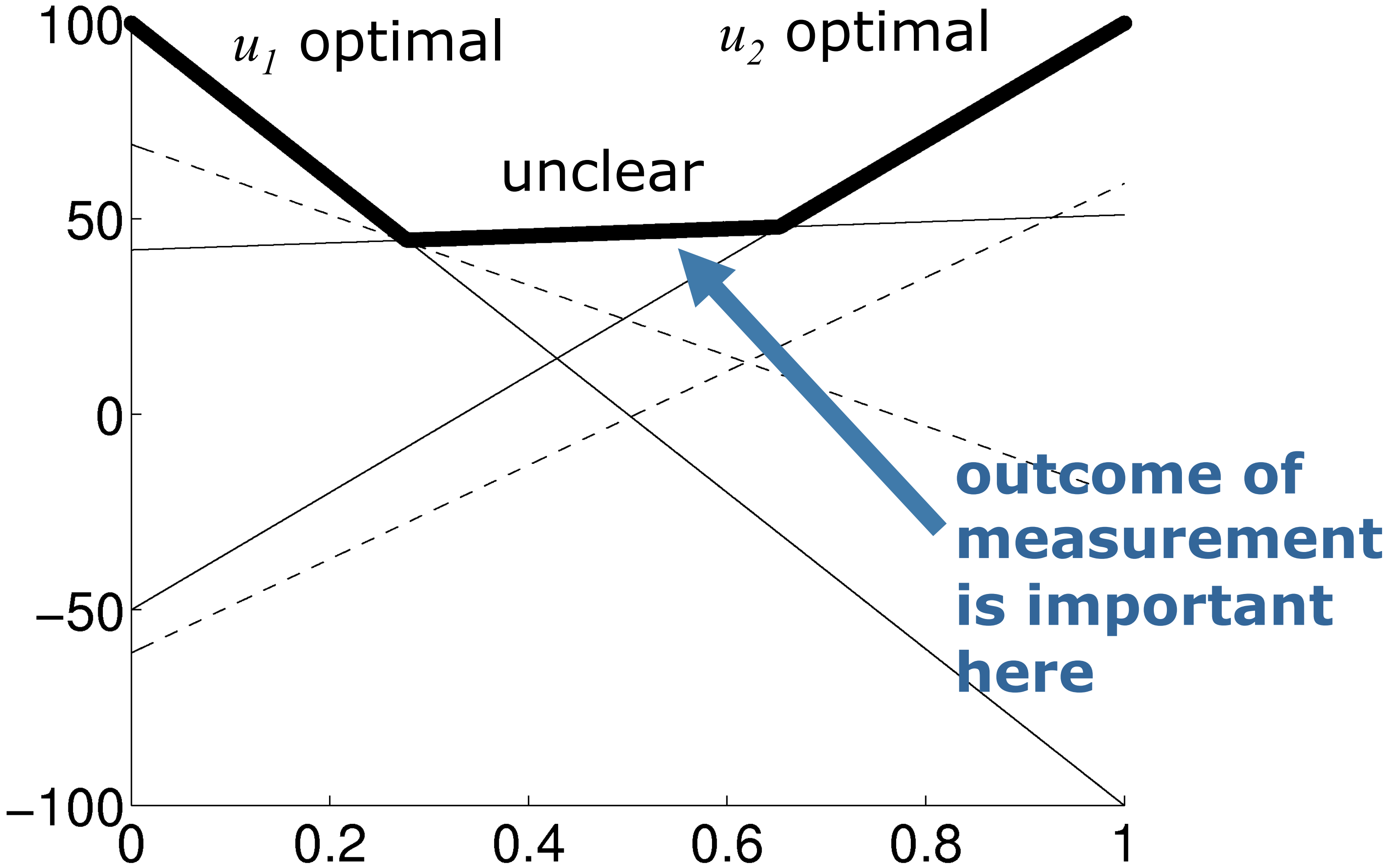
$$\bar{V}_1(b \mid u_3) \;=\; \max \left\{ \begin{array}{rr} 60\, p_1 & -60\,(1-p_1) \\ 52\, p_1 & +43\,(1-p_1) \\ -20\, p_1 & +70\,(1-p_1) \end{array} \right\}$$

Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2, or first u_3 and then u_1 or u_2, we obtain (after pruning)

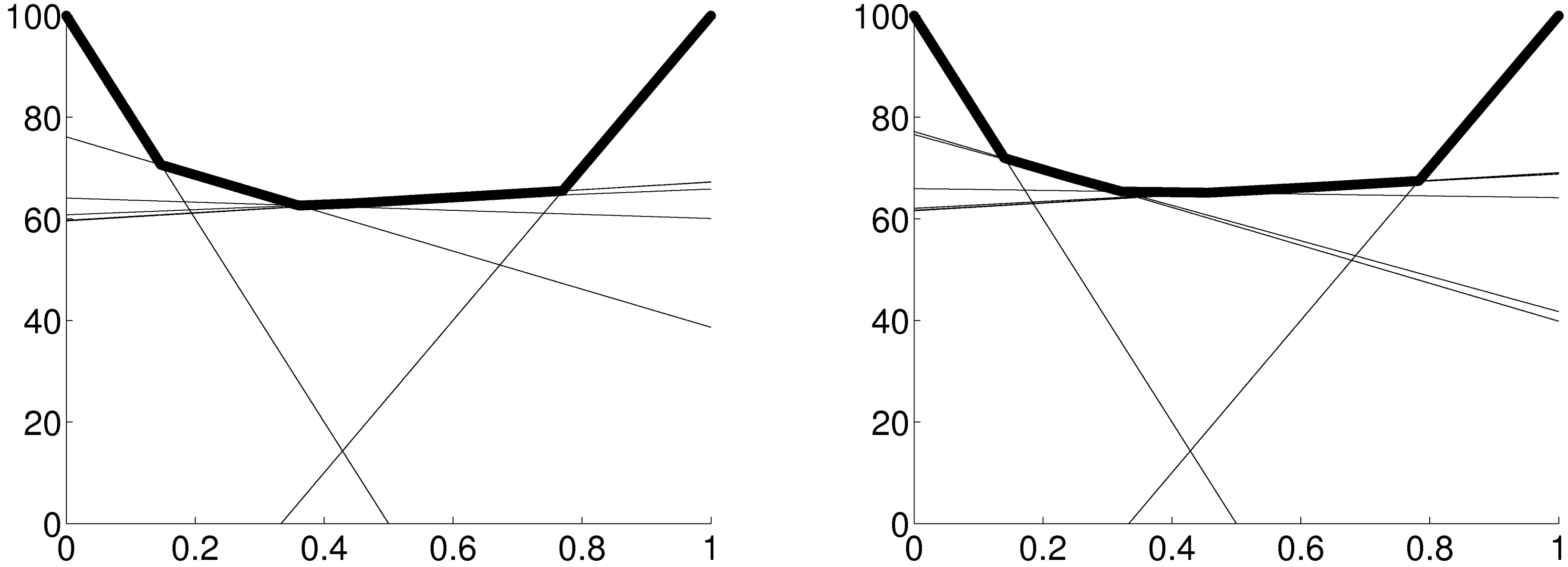
$$\bar{V}_2(b) \;=\; \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ 51\, p_1 & +42\,(1-p_1) \end{array} \right\}$$

Graphical Representation of $V_2(b)$



Deep Horizons and Pruning

- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are



Why Pruning is Essential

- Each **update introduces additional linear components** to V.
- Each **measurement squares the number of linear components**.
- Thus, an unpruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why **POMDPs are impractical for most applications**.

Summary on POMDPs

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.