Introduction to Mobile Robotics

Bayes Filter – Particle Filter and Monte Carlo Localization

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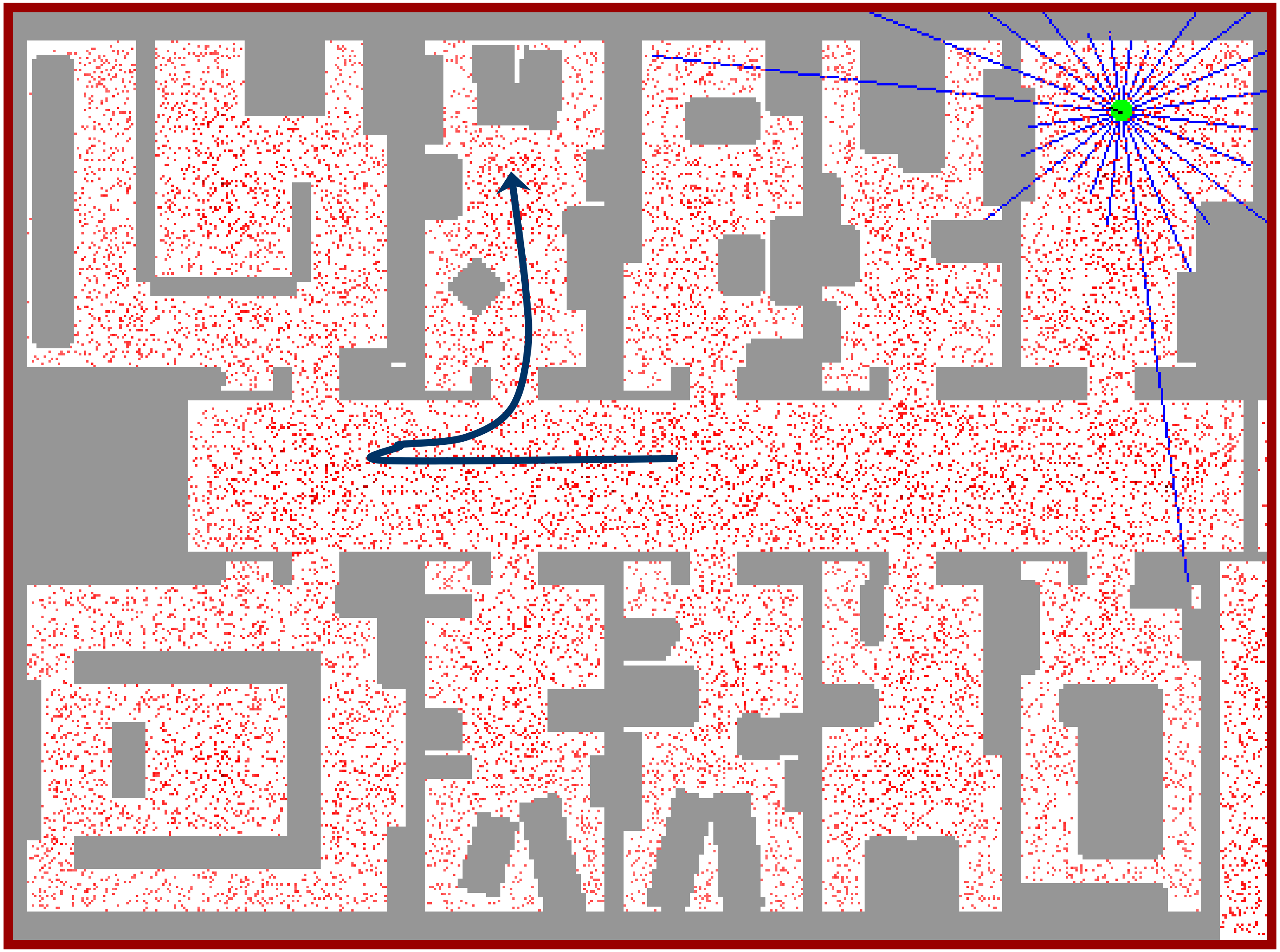
Motivation

- Recall: Discrete filter
  - Discretize the continuous state space
  - High memory complexity
  - Fixed resolution (does not adapt to the belief)

- Particle filters are a way to **efficiently** represent **non-Gaussian distribution**

- Basic principle
  - Set of state hypotheses (“particles”)
  - Survival-of-the-fittest

Sample-based Localization (sonar)



Mathematical Description

- Set of weighted samples

$$S = \left\{ \left\langle s^{[i]}, w^{[i]} \right\rangle \mid i = 1, \ldots, N \right\}$$

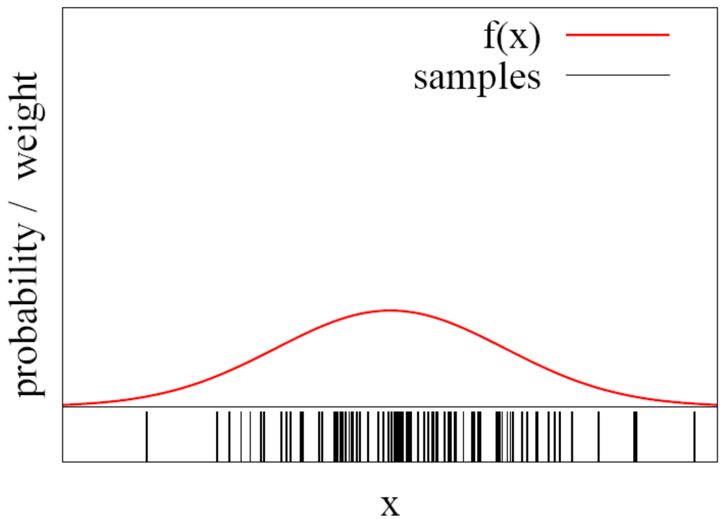
State hypothesis   Importance weight

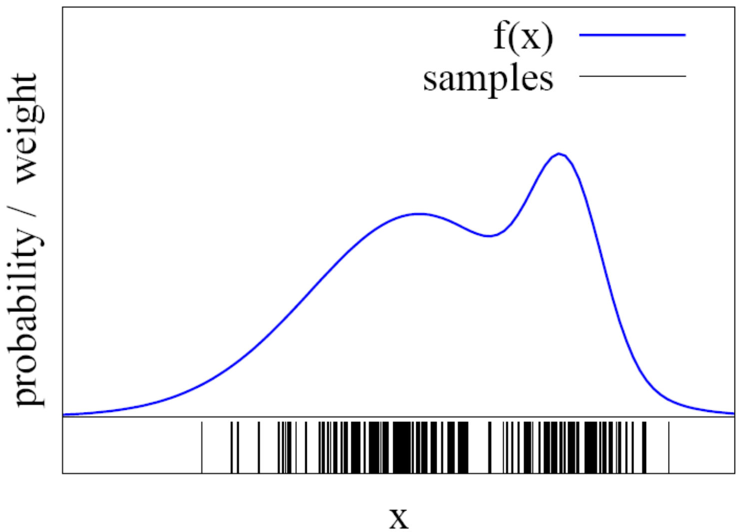
- The samples represent the posterior

$$p(x) = \sum_{i=1}^{N} w_i \cdot \delta_{s^{[i]}}(x)$$

Function Approximation

- Particle sets can be used to approximate functions





- The more particles fall into an interval, the higher the probability of that interval

- How to draw samples from a function/distribution?

Bayes filter with particle sets

- Measurement update

$$bel(x) \leftarrow p(z|x)\overline{bel}(x)$$

$$= p(z|x) \sum_i w_i \, \delta_{s^{[i]}}(x) = \sum_i p\left(z \middle| s^{[i]}\right) w_i \, \delta_{s^{[i]}}(x)$$

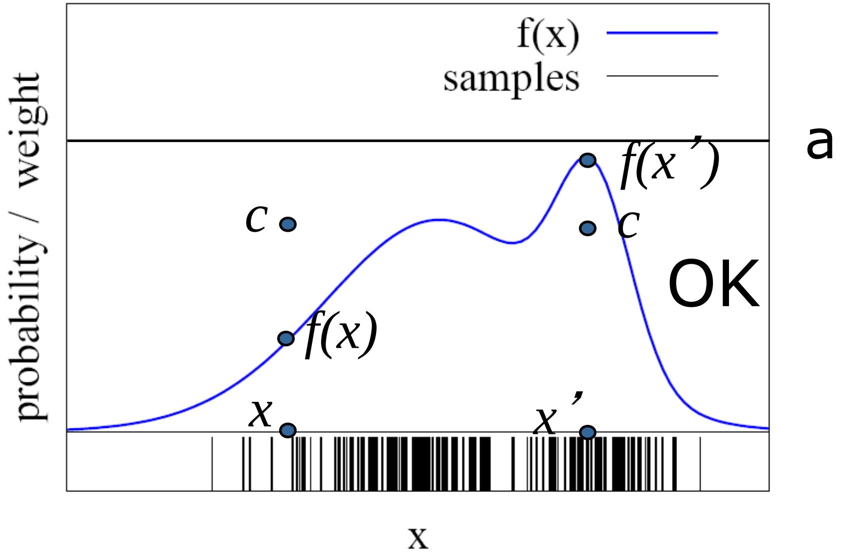
- Motion update

$$\overline{bel}(x) \leftarrow \int p(x|u, x^-) \, \mathrm{bel}(x^-) dx^-$$

$$= \int p(x|u, x^-) \sum_i w_i \, \delta_{s^{[i]}}(x^-) dx^- = \sum_i p\left(x \middle| u, s^{[i]}\right) w_i$$

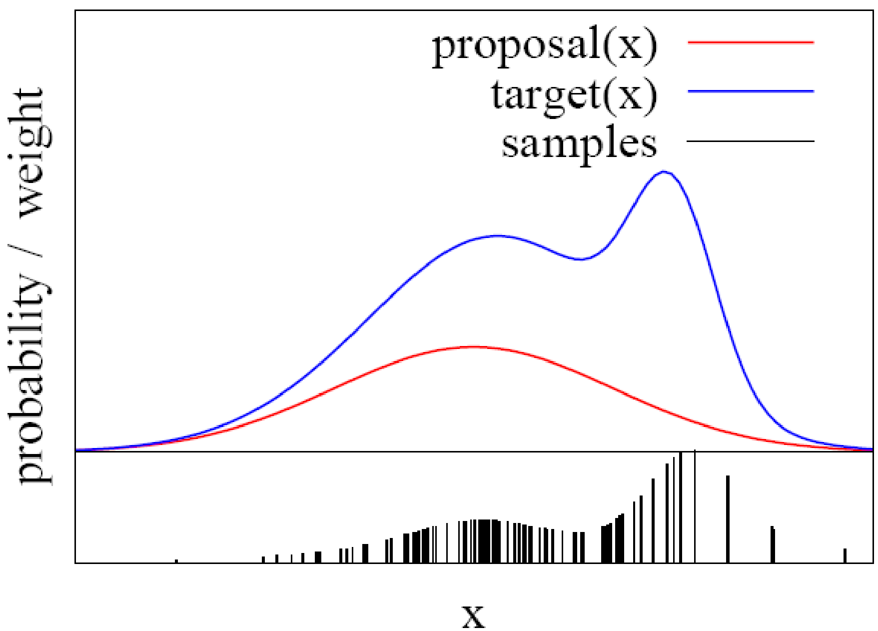
Rejection Sampling

- Let us assume that $f(x) < a$ for all x
- Sample x from a uniform distribution
- Sample c from [0,a]
- if $f(x) > c$ keep the sample
  otherwise reject the sample



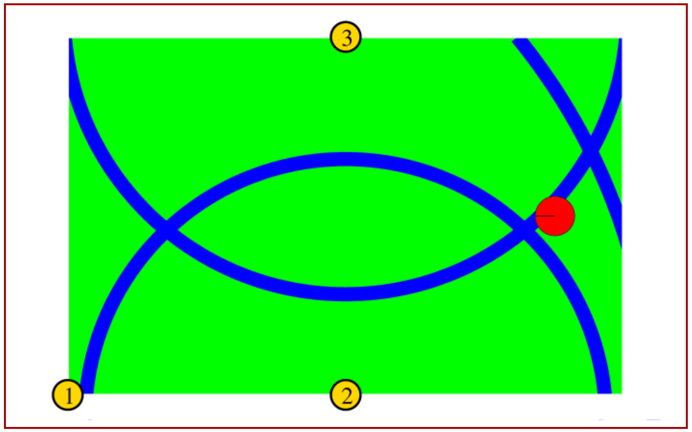
Importance Sampling Principle

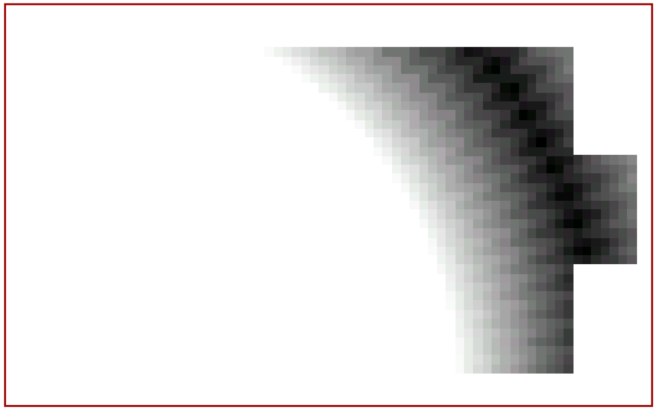
- We can even use a different distribution g to generate samples from f
- By introducing an importance weight w, we can account for the "differences between g and f"
- $w = f / g$
- f is called target
- g is called proposal
- Pre-condition: $f(x)>0 \rightarrow g(x)>0$
- Derivation: See webpage

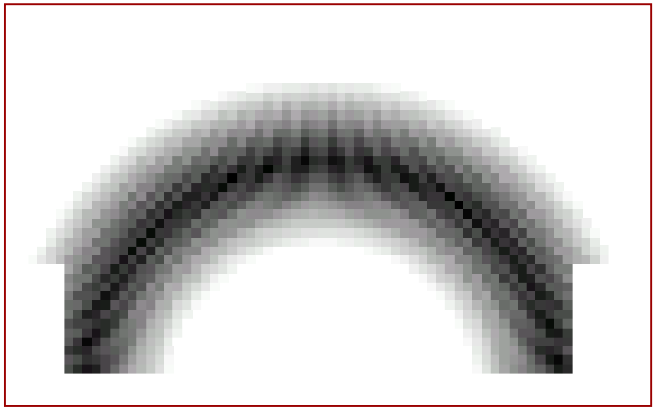


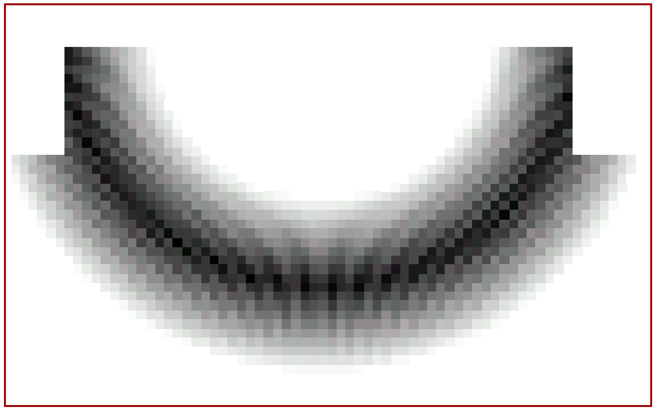
Importance Sampling with Resampling: Landmark Detection Example

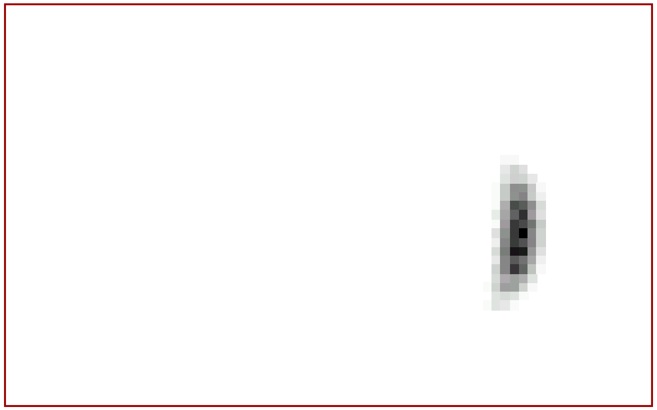
Distributions



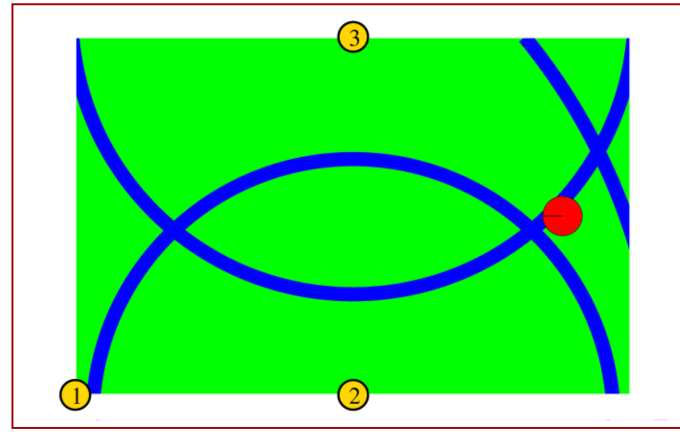


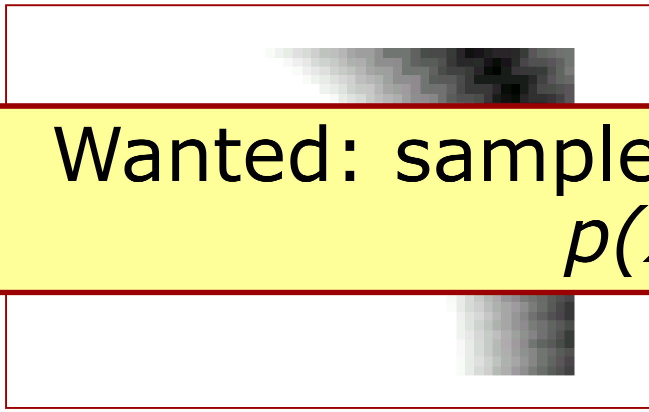


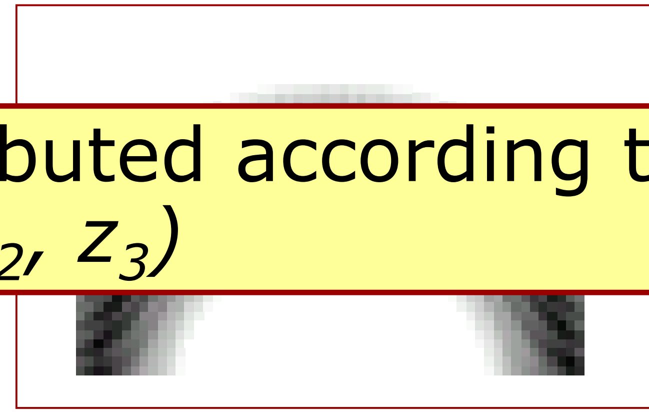




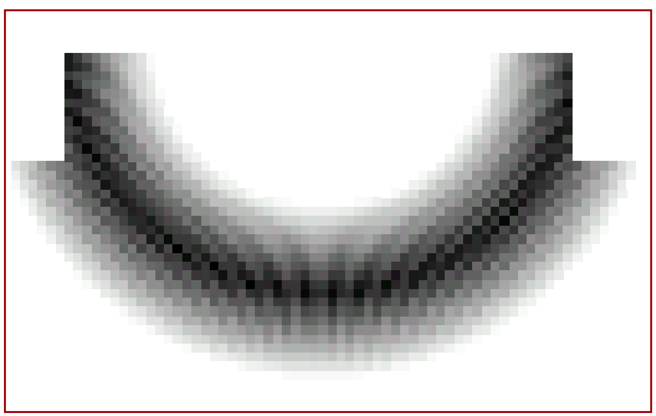
Distributions

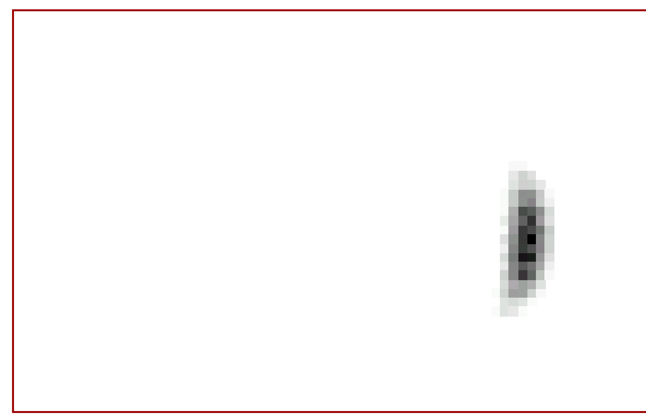






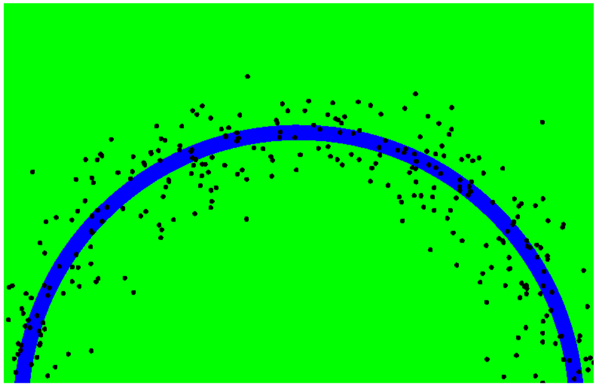
Wanted: samples distributed according to $p(x|\, z_1, z_2, z_3)$

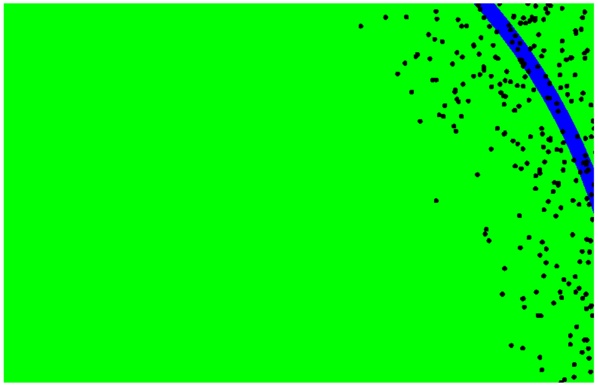


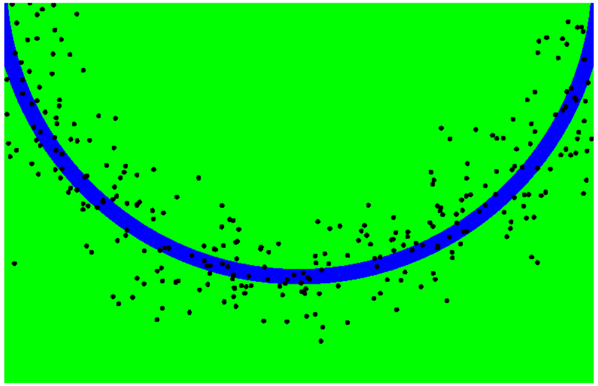


This is Easy!

We can draw samples from $p(x|z_l)$ by adding noise to the detection parameters.







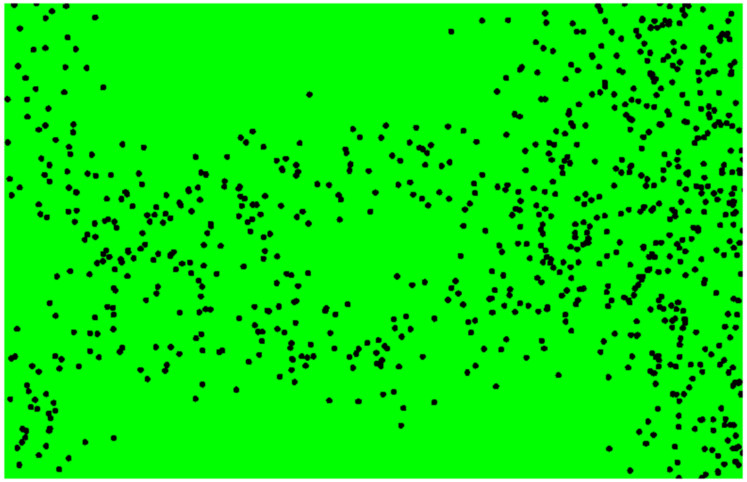
Importance Sampling

$$\text{Target distribution f}: p(x \mid z_1, z_2, \ldots, z_n) = \frac{\prod_k p(z_k \mid x) \quad p(x)}{p(z_1, z_2, \ldots, z_n)}$$

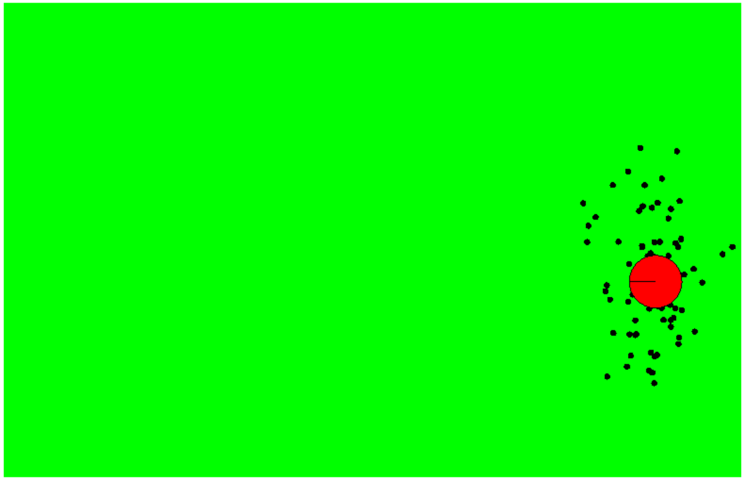
$$\text{Sampling distribution g: } p(x \mid z_l) = \frac{p(z_l \mid x) p(x)}{p(z_l)}$$

$$\text{Importance weights w: } \frac{f}{g} = \frac{p(x \mid z_1, z_2, \ldots, z_n)}{p(x \mid z_l)} = \frac{p(z_l) \prod_{k \neq l} p(z_k \mid x)}{p(z_1, z_2, \ldots, z_n)}$$

Importance Sampling with Resampling

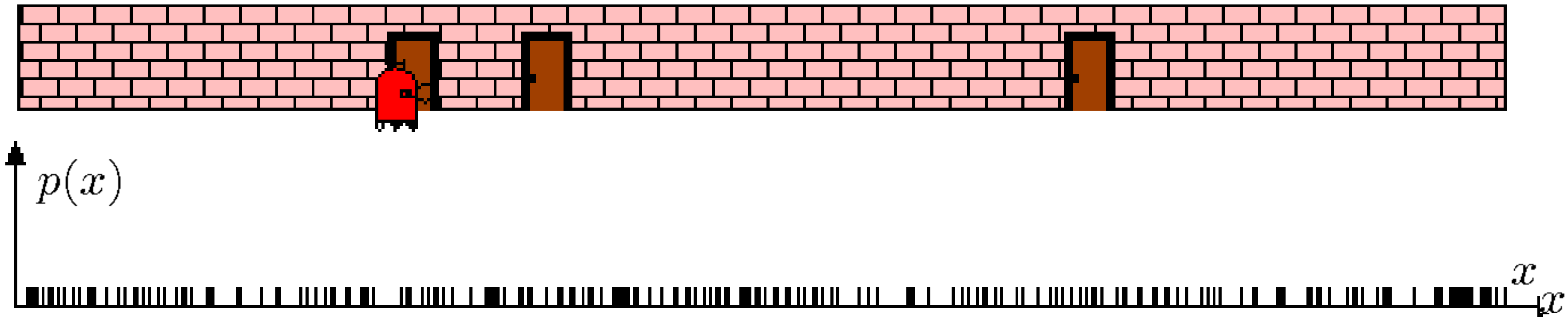


Weighted samples



After resampling

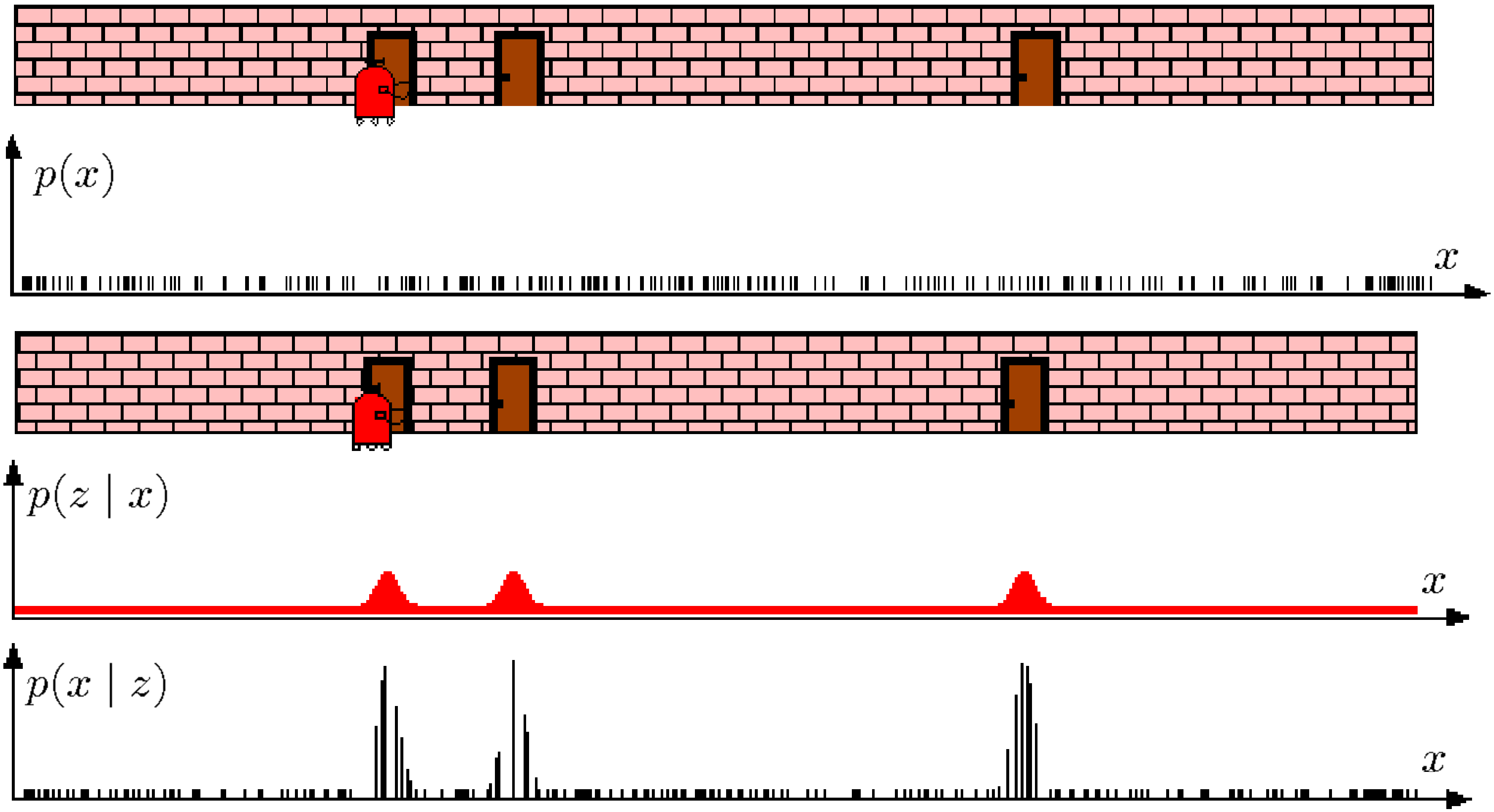
Particle Filters



Sensor Information: Importance Sampling

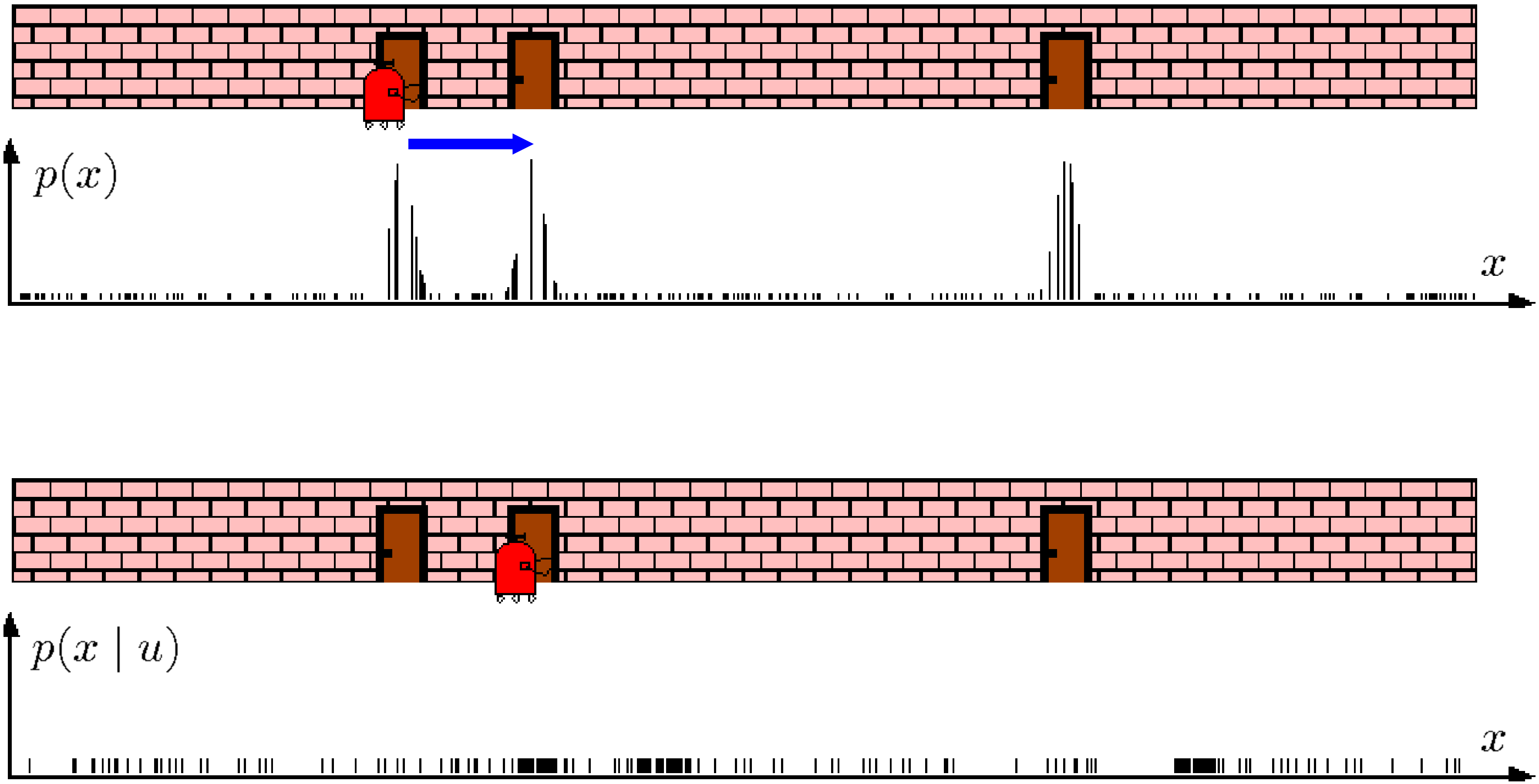
$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^-(x)$$

$$w \leftarrow \frac{\alpha\, p(z \mid x)\, Bel^-(x)}{Bel^-(x)} = \alpha\, p(z \mid x)$$



Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\, \mathrm{d}x'$$



Sensor Information: Importance Sampling

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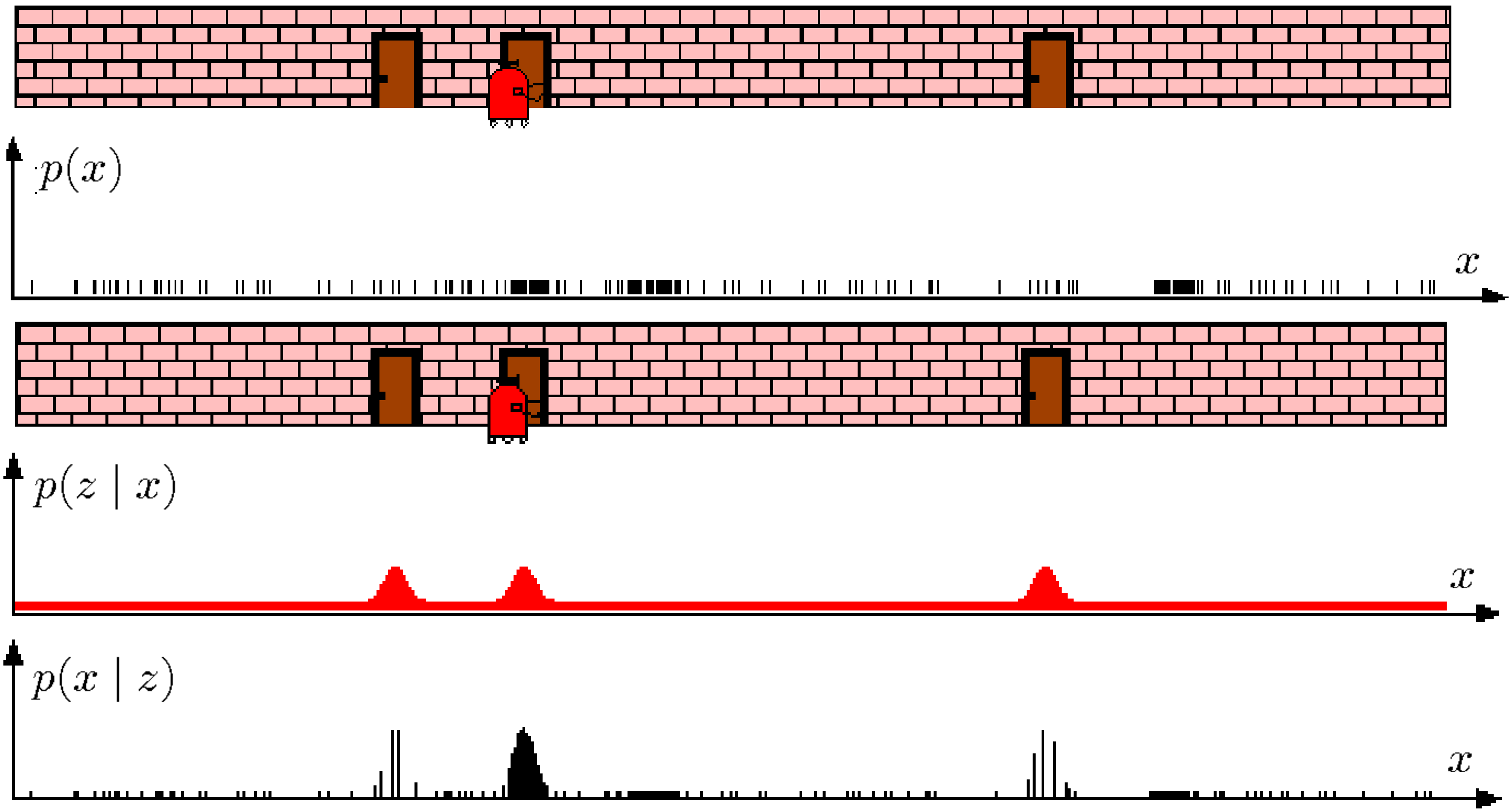

Robot Motion

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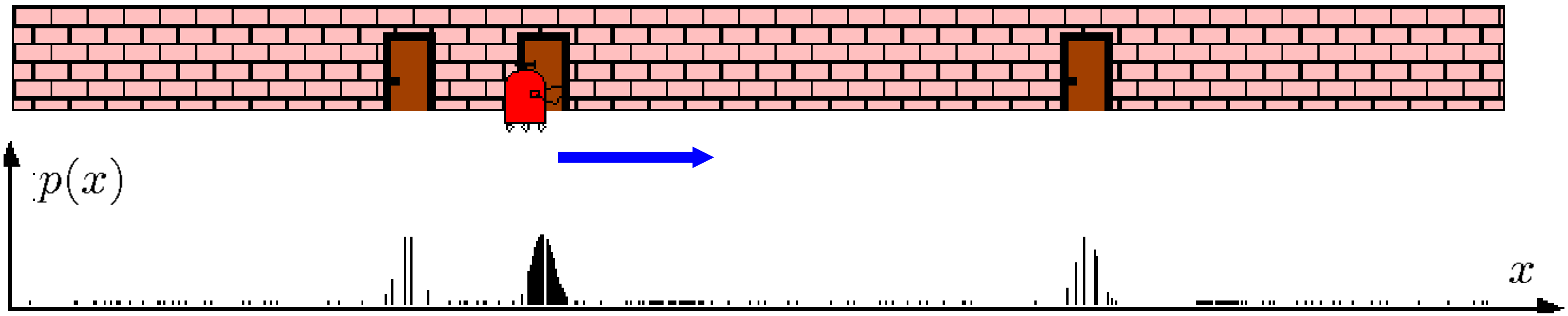
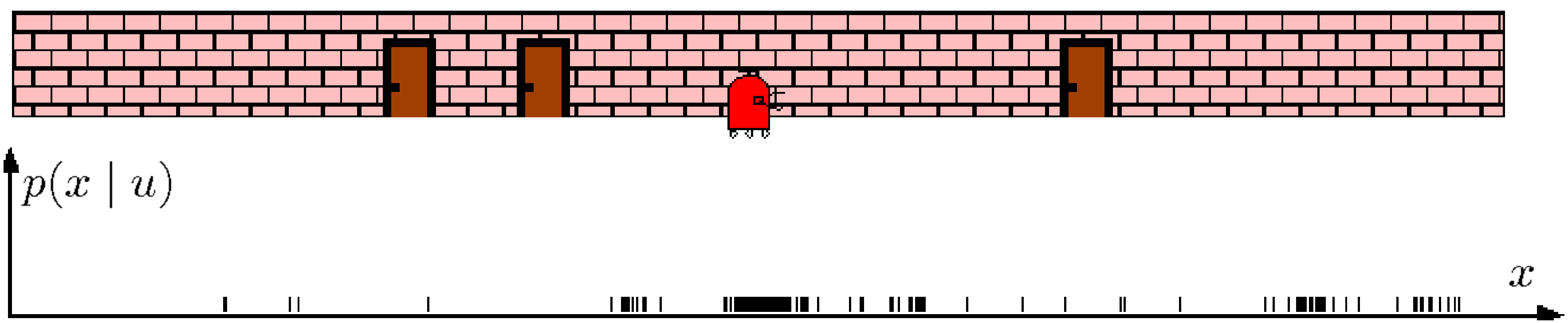




Particle Filter Algorithm

- Sample the next generation for particles using the proposal distribution

- Compute the importance weights :

$$weight = target\ distribution\ /\ proposal\ distribution$$

- Resampling: "Replace unlikely samples by more likely ones"

Particle Filter Algorithm

1. Algorithm **particle_filter**(S_{t-1}, u_t, z_t):
2. $S_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1, \ldots, n$ ***Generate new samples***
4. Sample index $j(i)$ from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_t)$ using $x_{t-1}^{j(i)}$ and u_t
6. $w_t^i = p(z_t \mid x_t^i)$ ***Compute importance weight***
7. $\eta = \eta + w_t^i$ ***Update normalization factor***
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ ***Add to new particle set***
9. **For** $i = 1, \ldots, n$
10. $w_t^i = w_t^i / \eta$ ***Normalize weights***

Particle Filter Algorithm

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, Bel(x_{t-1})\, dx_{t-1}$$

draw x^i_{t-1} from $Bel(x_{t-1})$

draw x^i_t from $p(x_t \mid x^i_{t-1}, u_t)$

Importance factor for x^i_t:

$$w^i_t = \frac{\text{target distribution}}{\text{proposal distribution}}$$

$$= \frac{\eta\, p(z_t \mid x_t)\, p(x_t \mid x_{t-1}, u_t)\, Bel\,(x_{t-1})}{p(x_t \mid x_{t-1}, u_t)\, Bel\,(x_{t-1})}$$

$$\propto p(z_t \mid x_t)$$

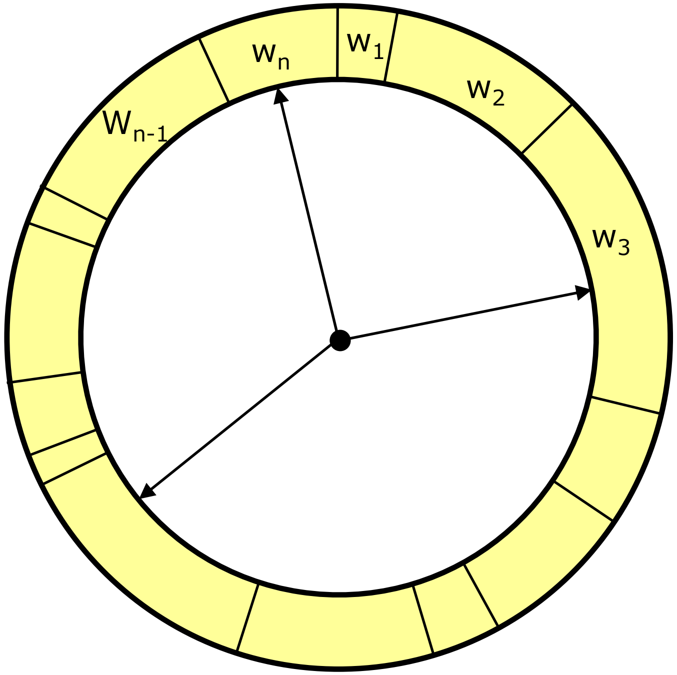
Resampling

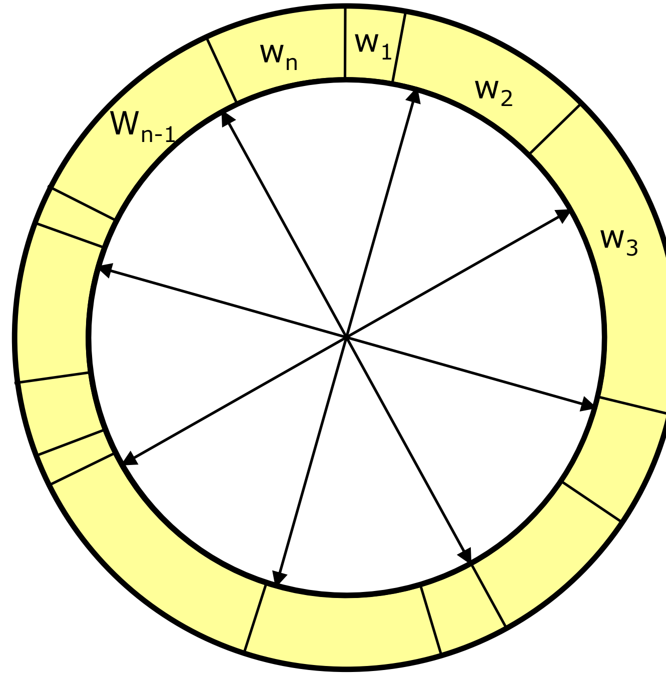
- **Given**: Set S of weighted samples.

- **Wanted** : Random sample, where the probability of drawing x_i is given by w_i.

- Typically done n times with replacement to generate new sample set S'.

Resampling





- Roulette wheel
- Binary search, n log n

- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

Resampling Algorithm

1. Algorithm **systematic_resampling**(*S*,*n*):

2. $S' = \emptyset, c_1 = w^1$
3. **For** $i = 2 \ldots n$ — ***Generate cdf***
4. $\quad c_i = c_{i-1} + w^i$
5. $u_1 \sim U\,]0, n^{-1}], i = 1$ — ***Initialize threshold***

6. **For** $j = 1 \ldots n$ — ***Draw samples …***
7. $\quad$ **While** ($u_j > c_i$) — ***Skip until next threshold reached***
8. $\quad\quad i = i + 1$
9. $\quad S' = S' \cup \{< x^i, n^{-1} >\}$ — ***Insert***
10. $\quad u_{j+1} = u_j + n^{-1}$ — ***Increment threshold***

11. **Return** S'

Also called **stochastic universal sampling**

Mobile Robot Localization

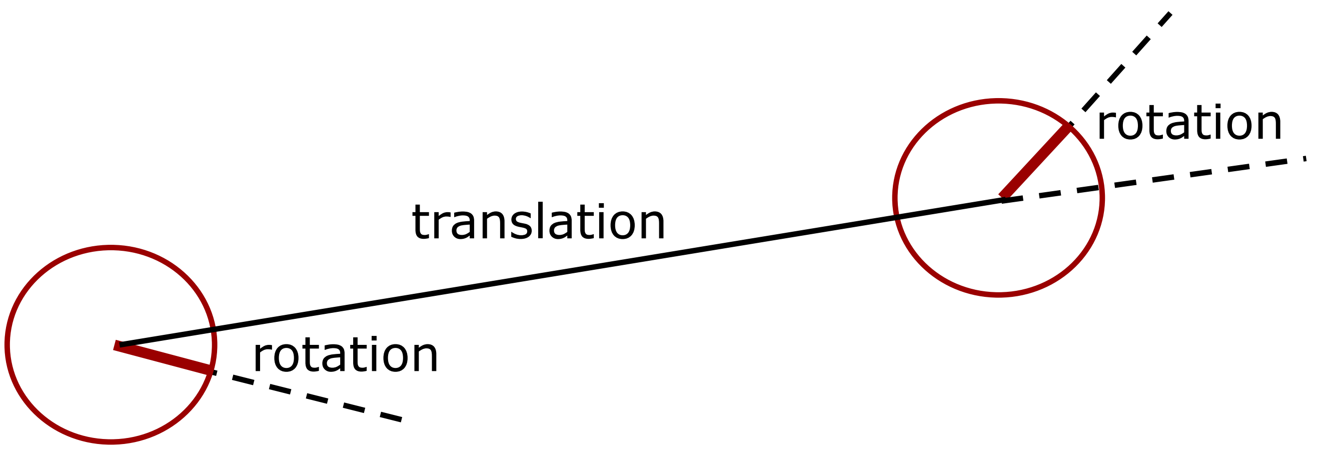
- Each particle is a potential pose of the robot
- Proposal distribution is the motion model of the robot (prediction step)
- The observation model is used to compute the importance weight (correction step)

[For details, see PDF file on the lecture web page]

Motion Model Reminder

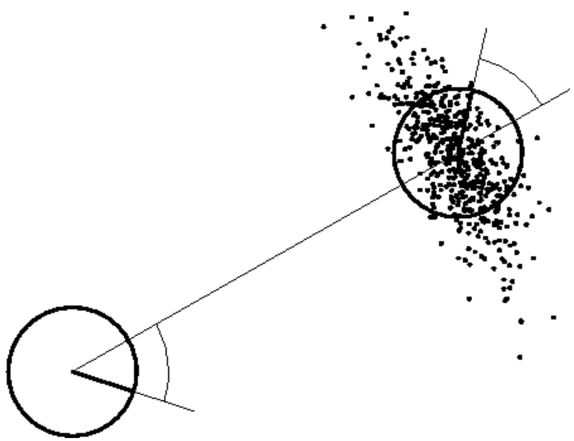
According to the estimated motion

Motion Model Reminder



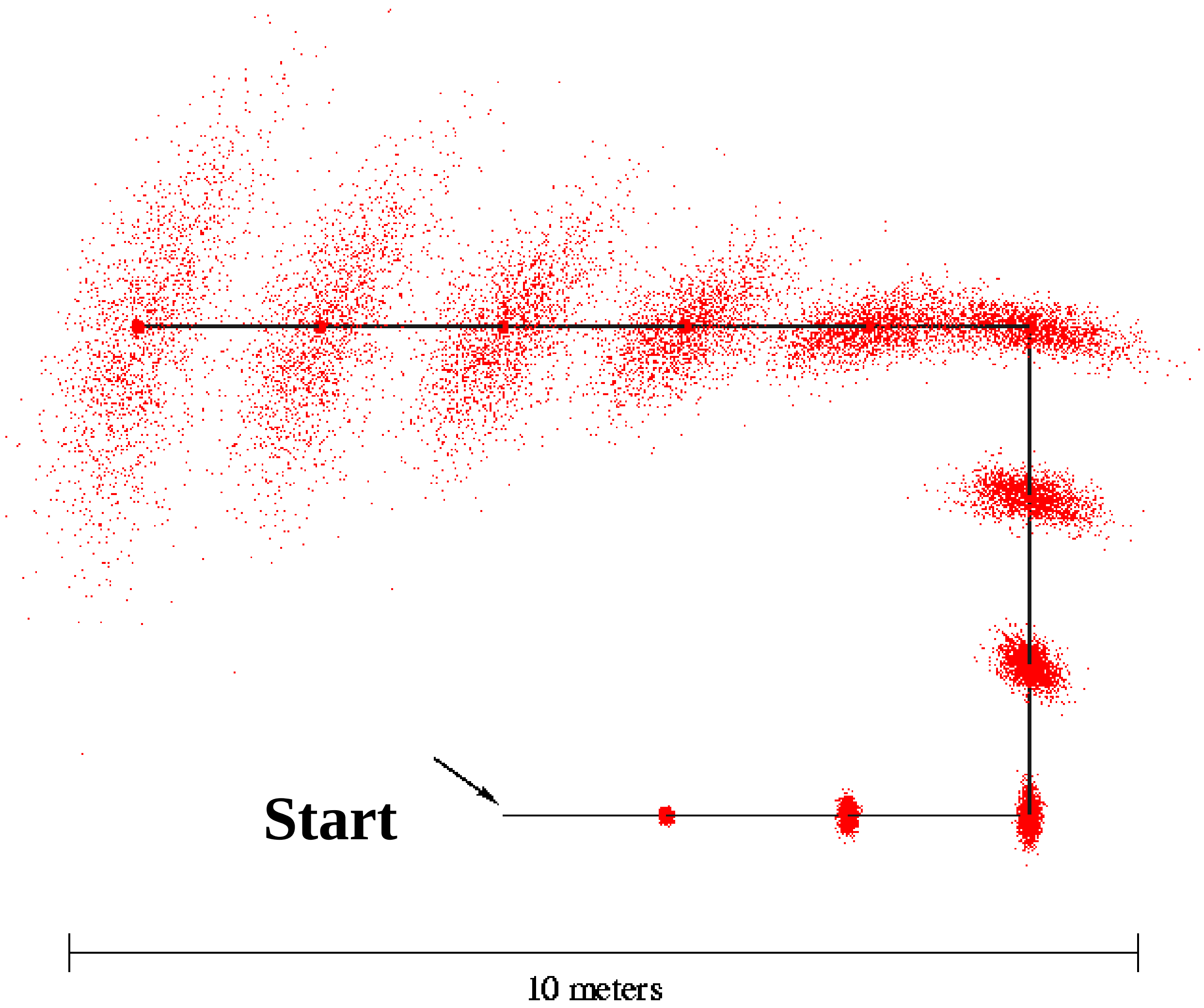
- Decompose the motion into
  - Traveled distance
  - Start rotation
  - End rotation

Motion Model Reminder

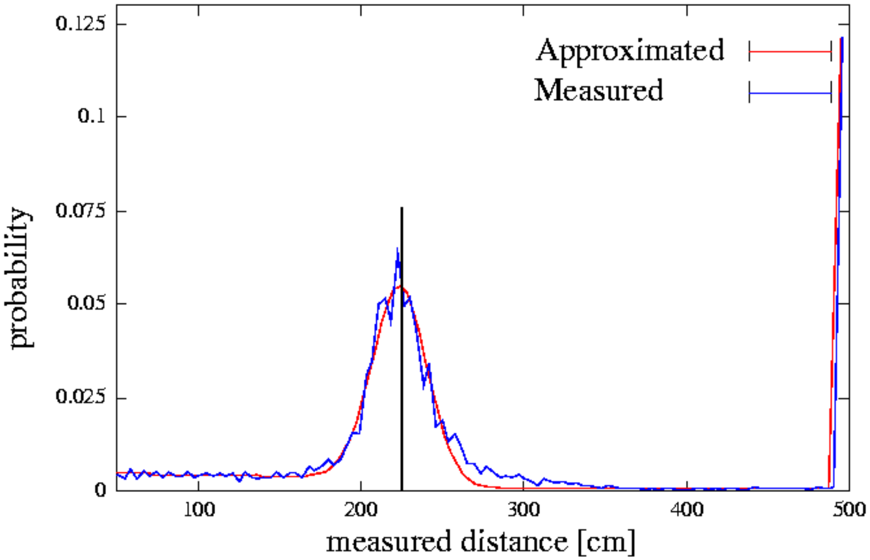


- Uncertainty in the translation of the robot: Gaussian over the traveled distance
- Uncertainty in the rotation of the robot: Gaussians over start and end rotation
- For each particle, draw a new pose by sampling from these three individual normal distributions

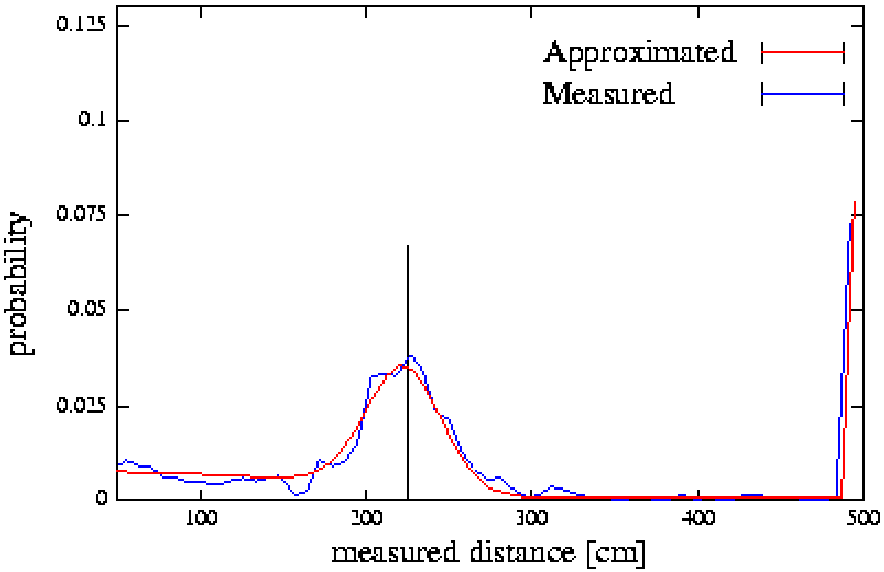
Motion Model Reminder



Proximity Sensor Model Reminder



Laser sensor



Sonar sensor

Mobile Robot Localization Using Particle Filters (1)

- Each particle is a potential pose of the robot
- The set of weighted particles approximates the posterior belief about the robot's pose (target distribution)

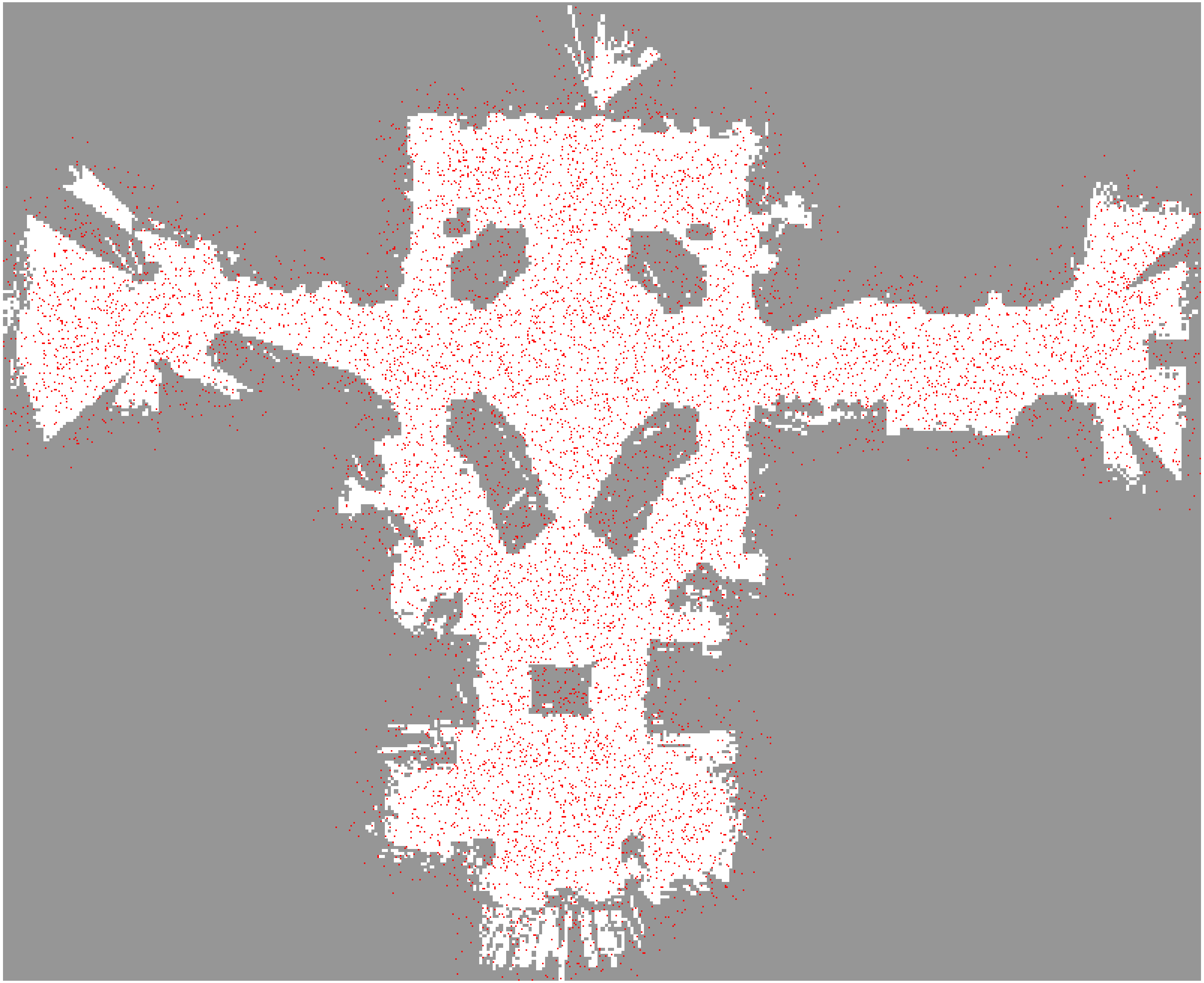
Mobile Robot Localization Using Particle Filters (2)

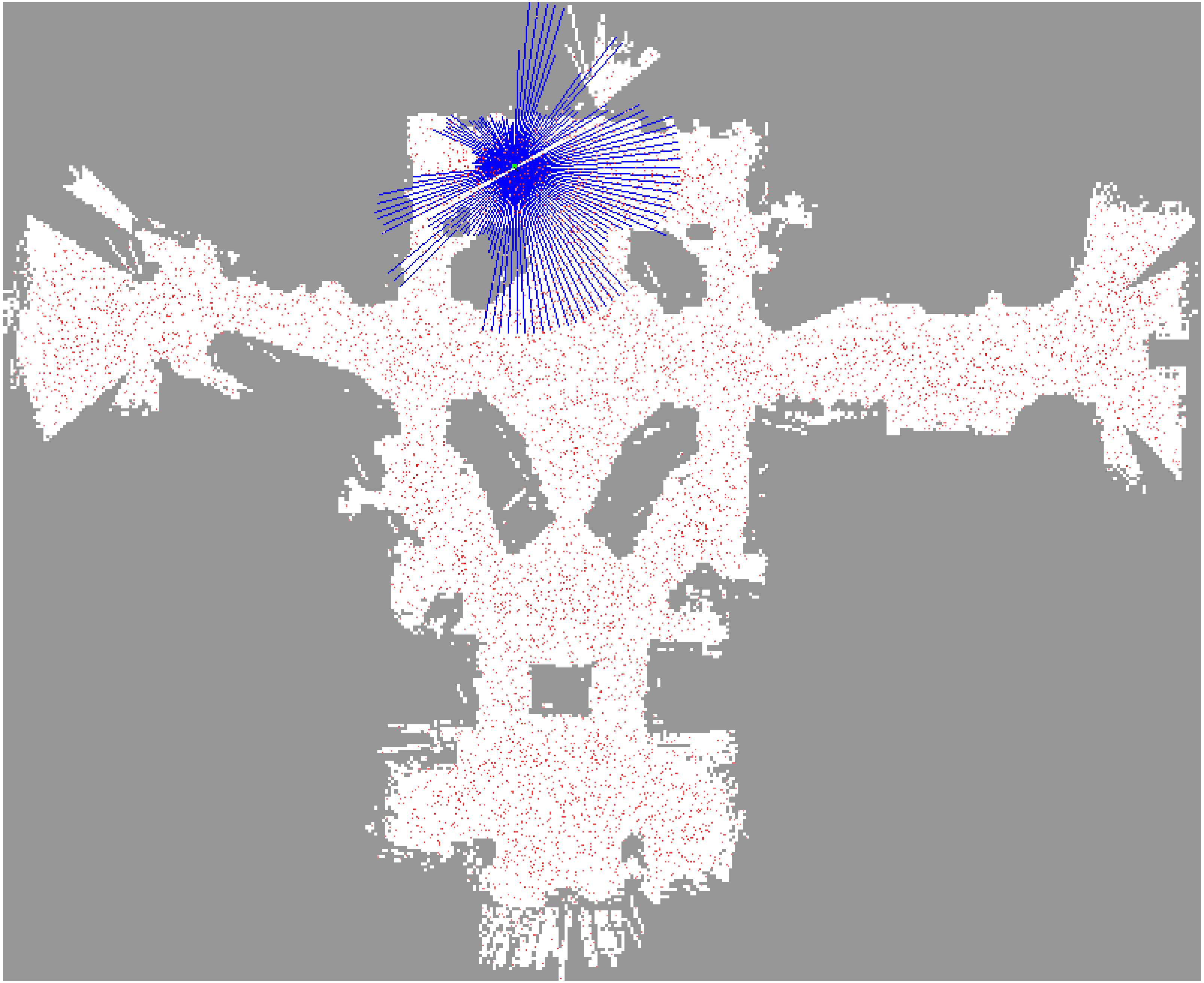
- Particles are drawn from the motion model (proposal distribution)
- Particles are weighted according to the observation model (sensor model)
- Particles are resampled according to the particle weights

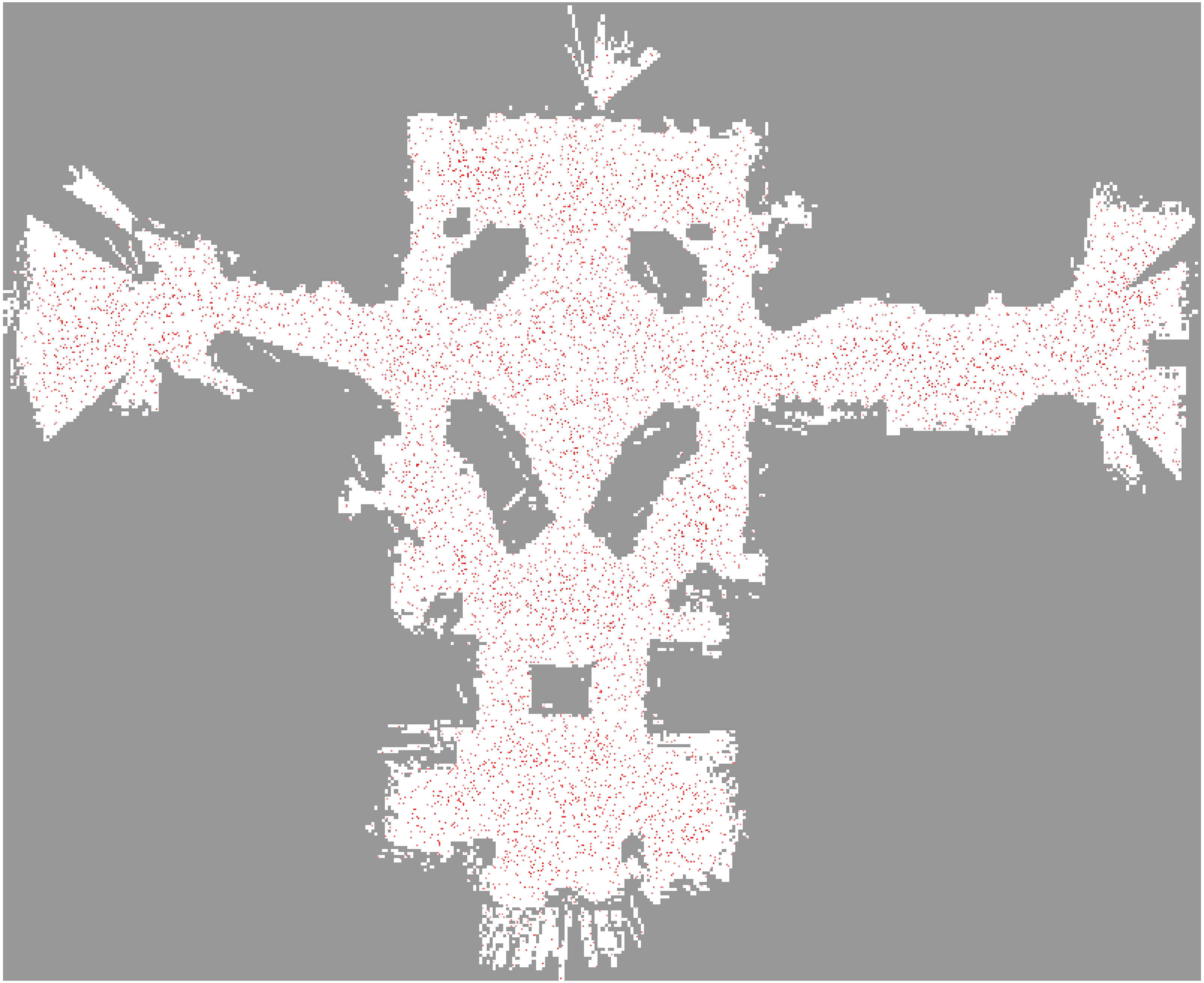
Mobile Robot Localization Using Particle Filters (3)

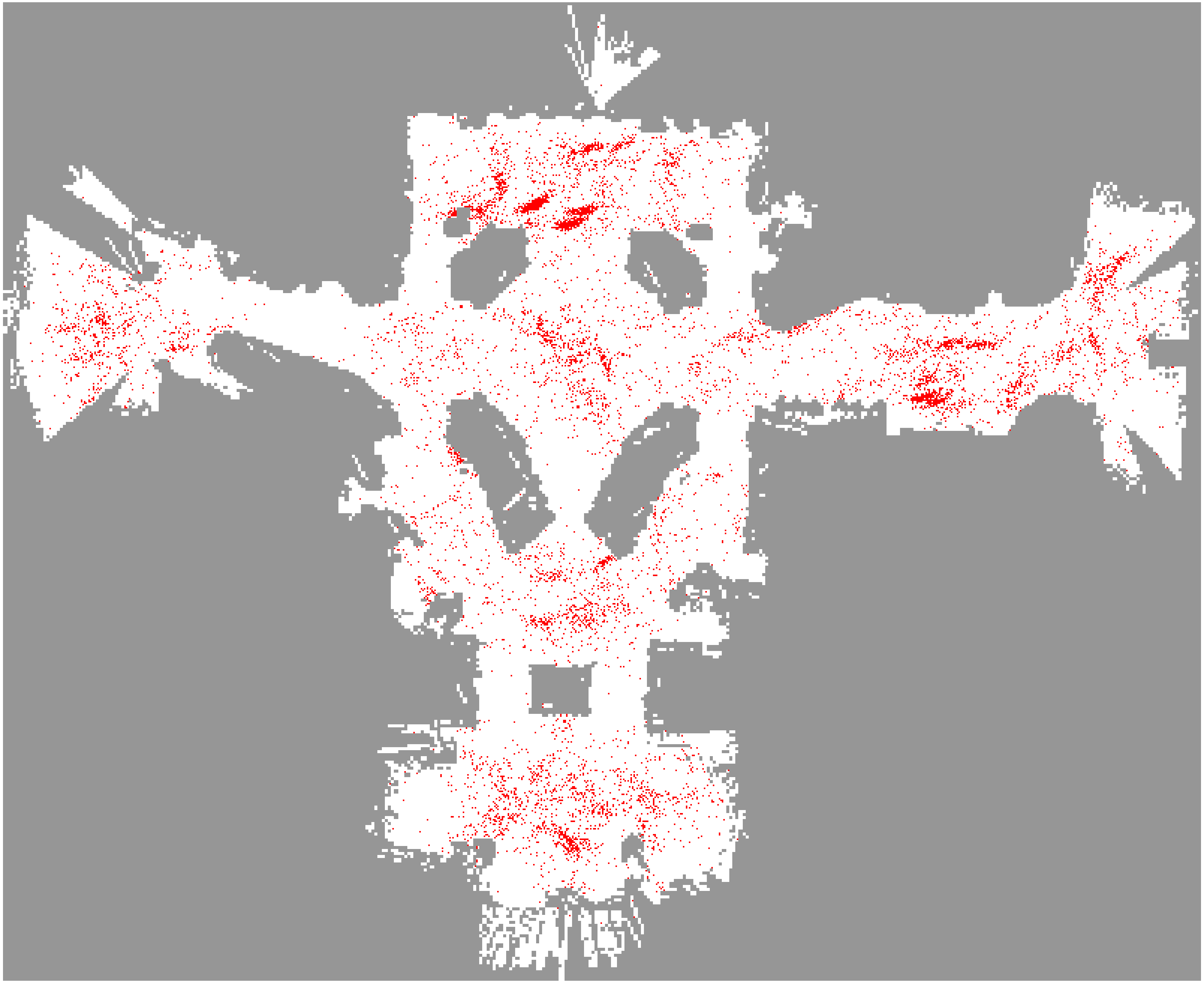
Why is resampling needed?

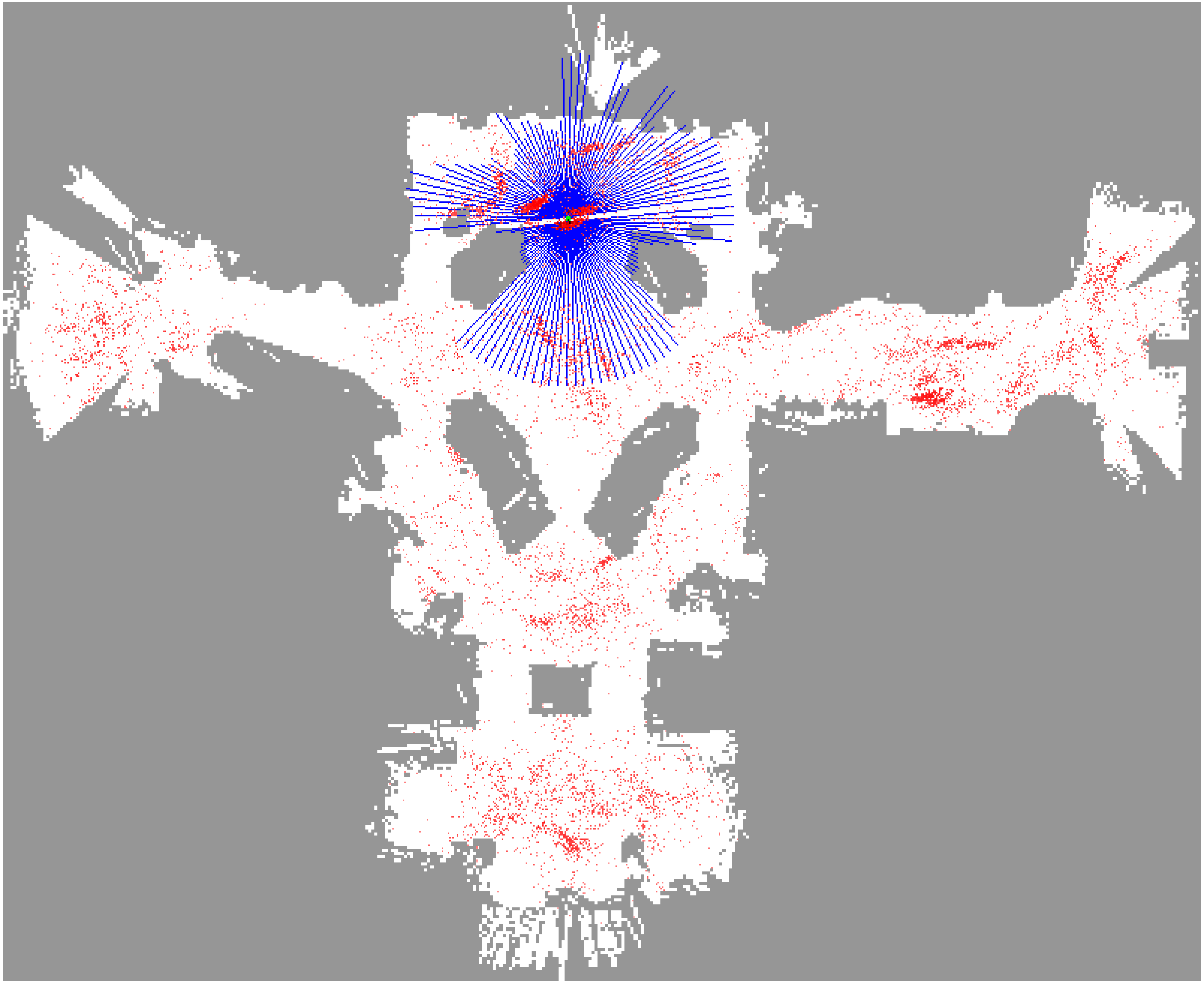
- We only have a finite number of particles
- Without resampling: The filter is likely to loose track of the “good” hypotheses
- Resampling ensures that particles stay in the meaningful area of the state space

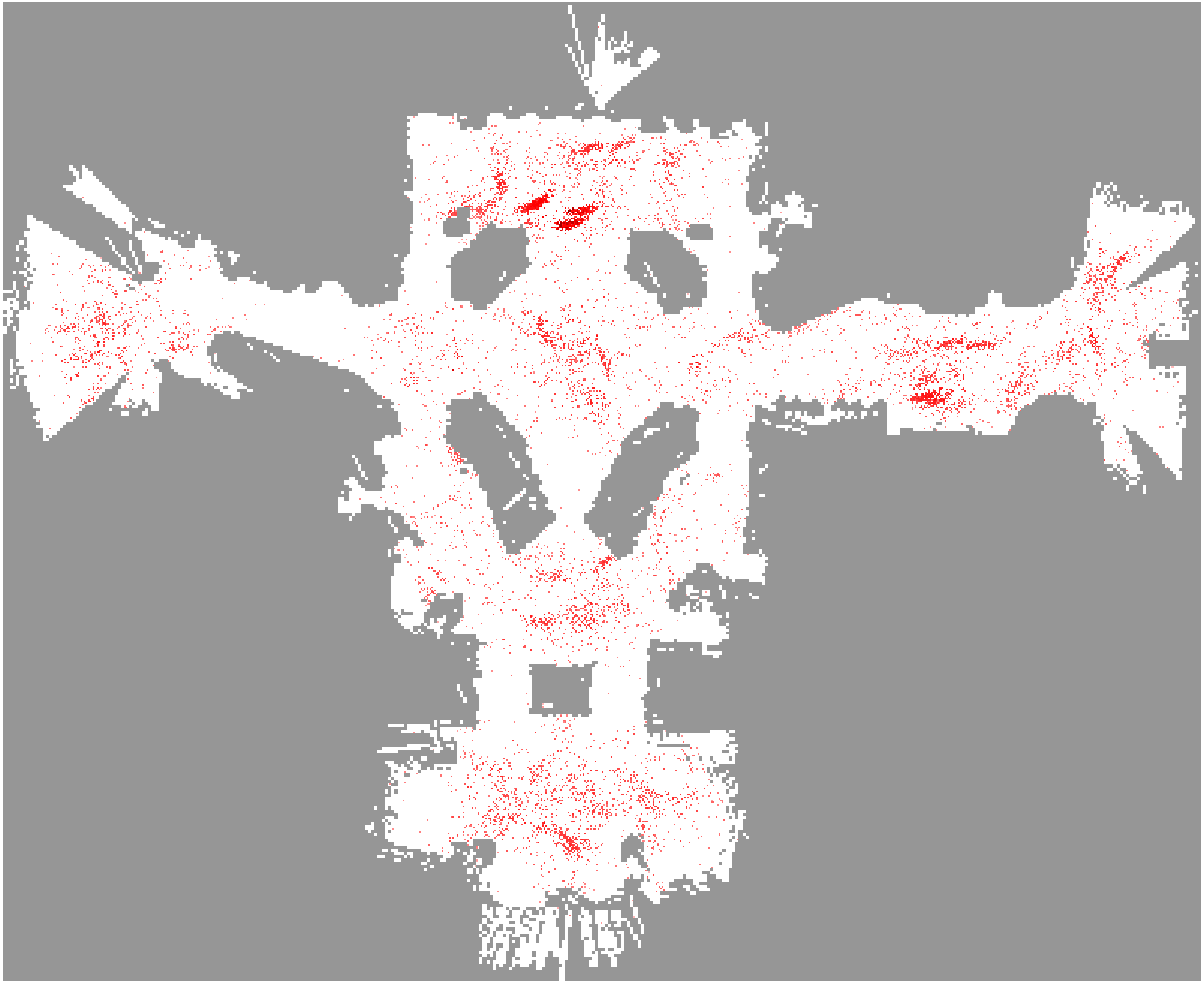


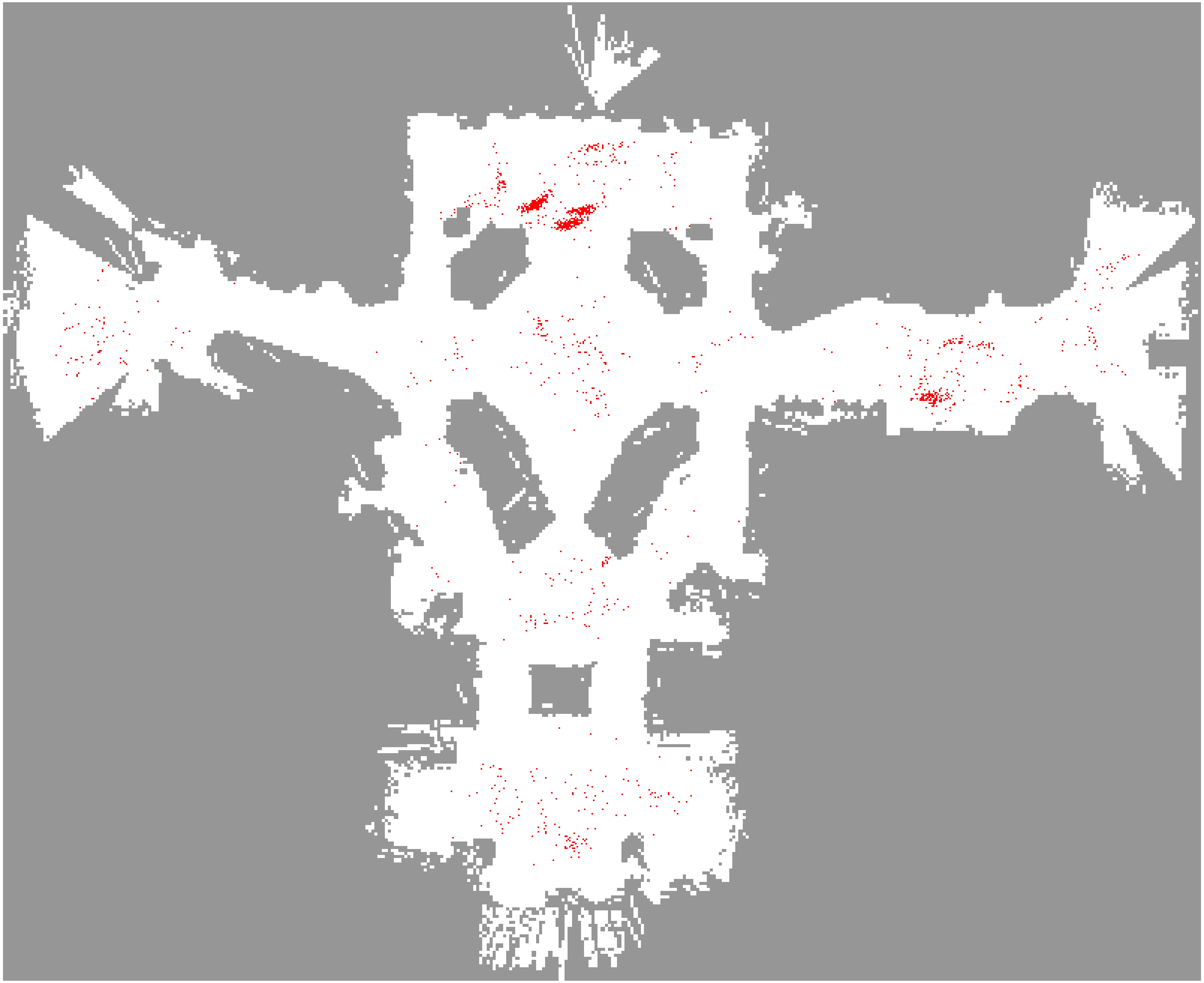


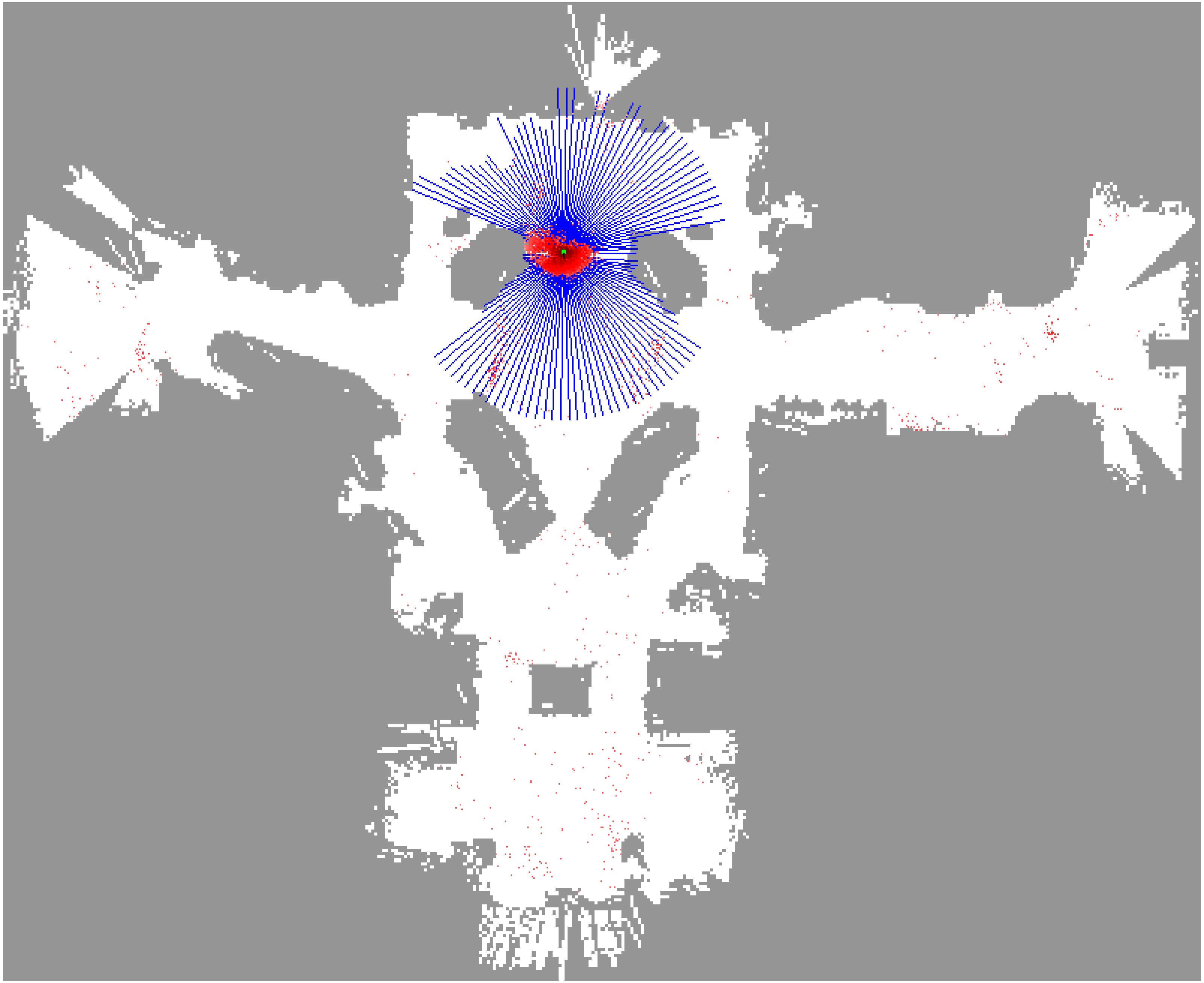


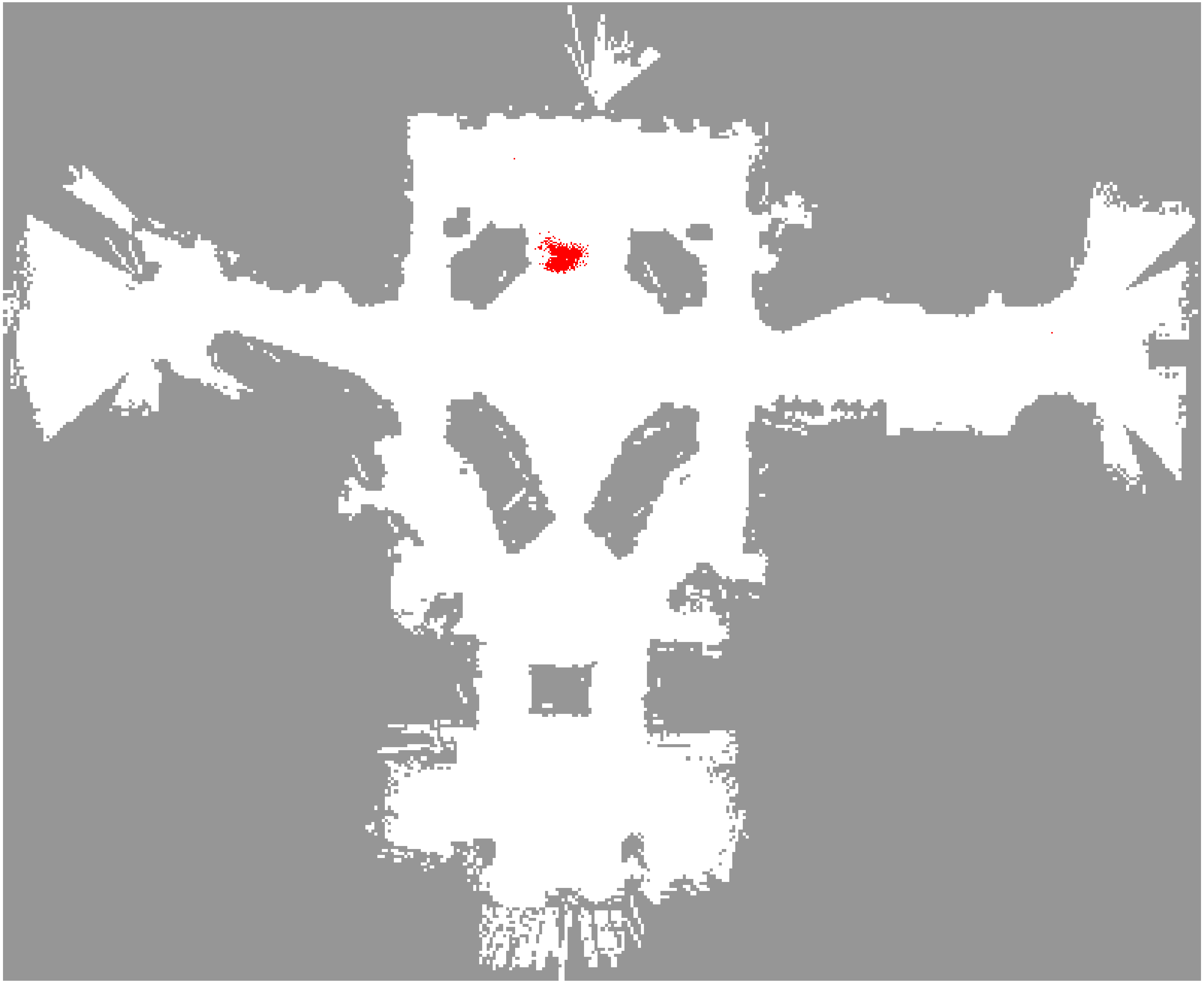


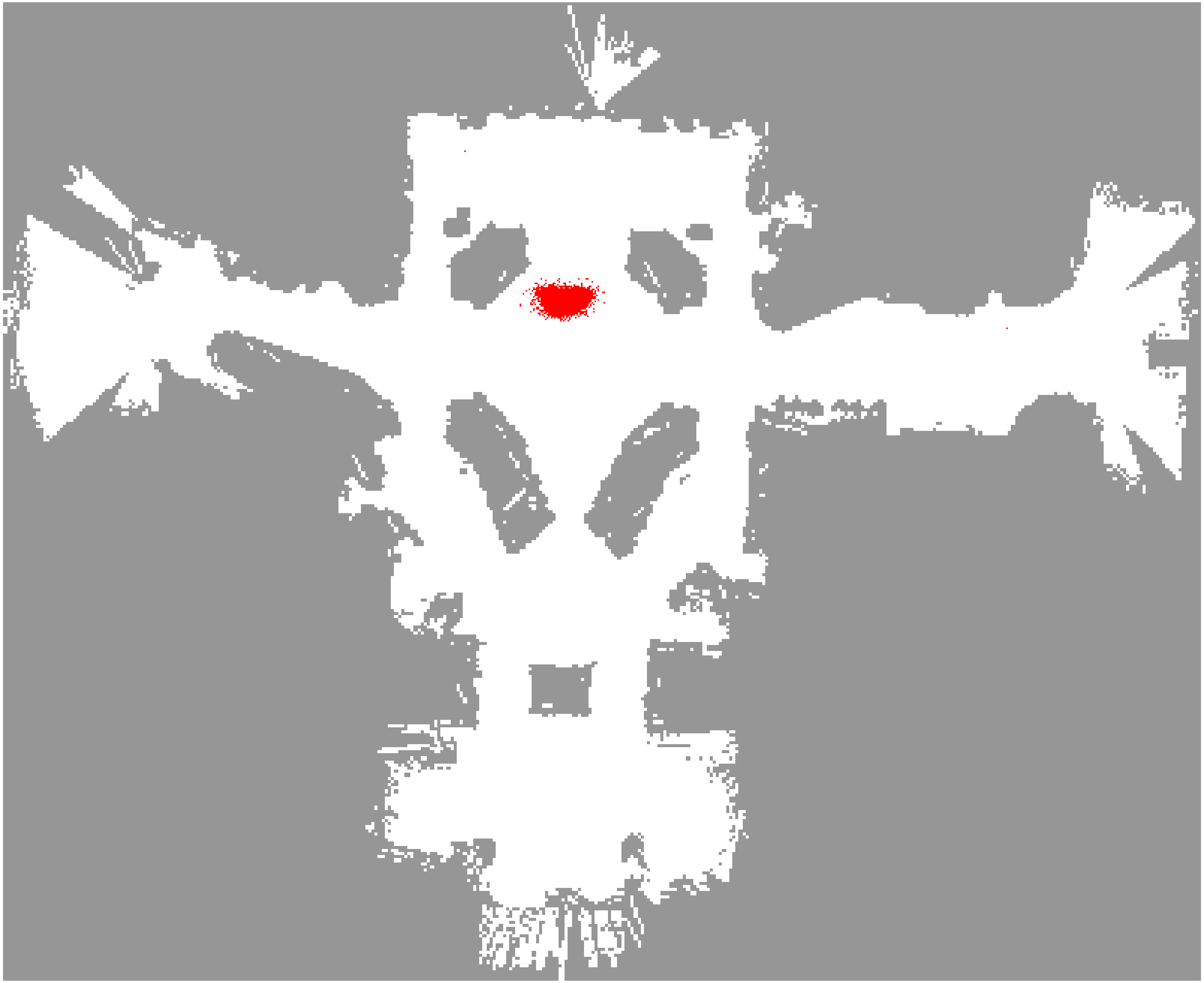


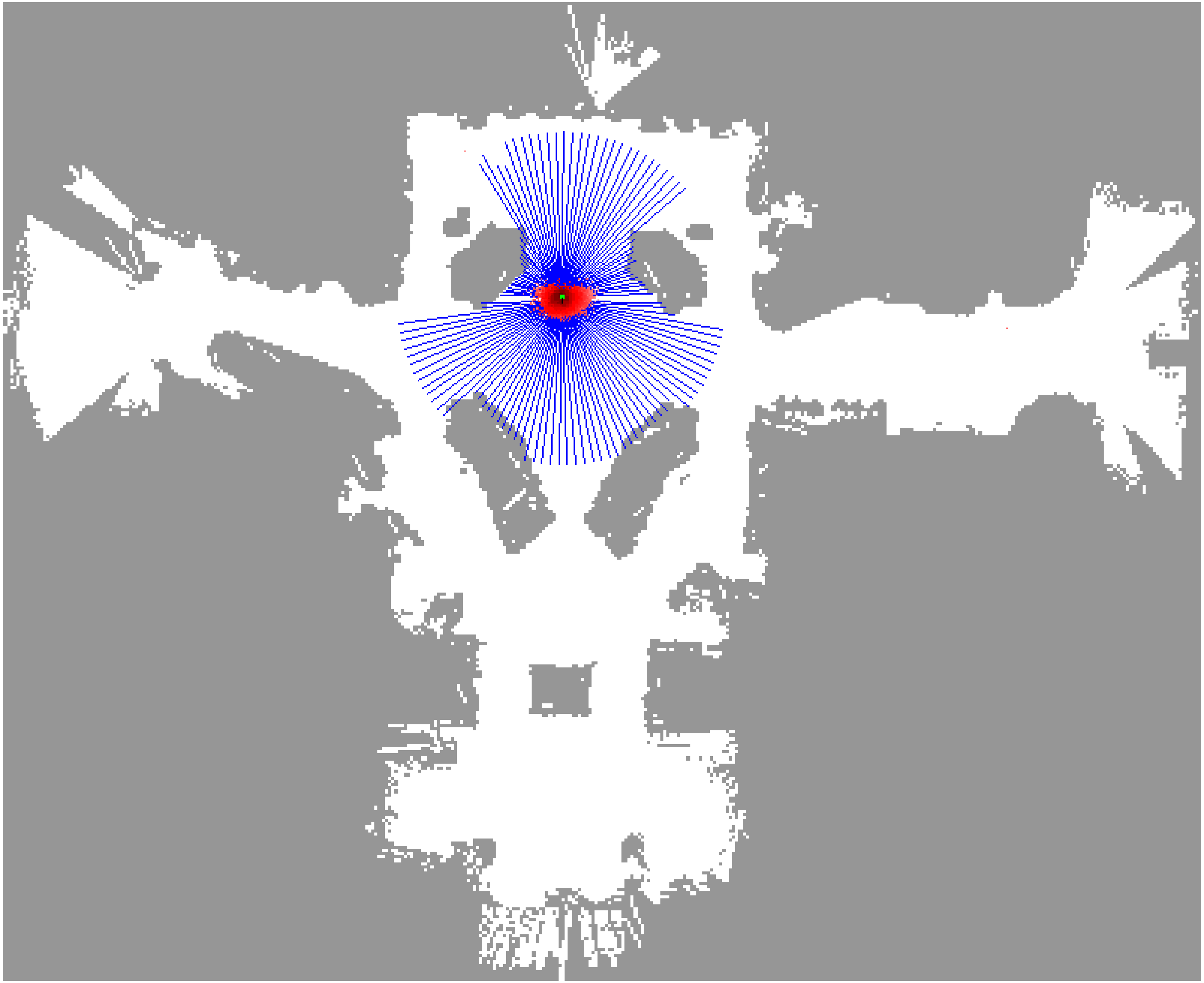


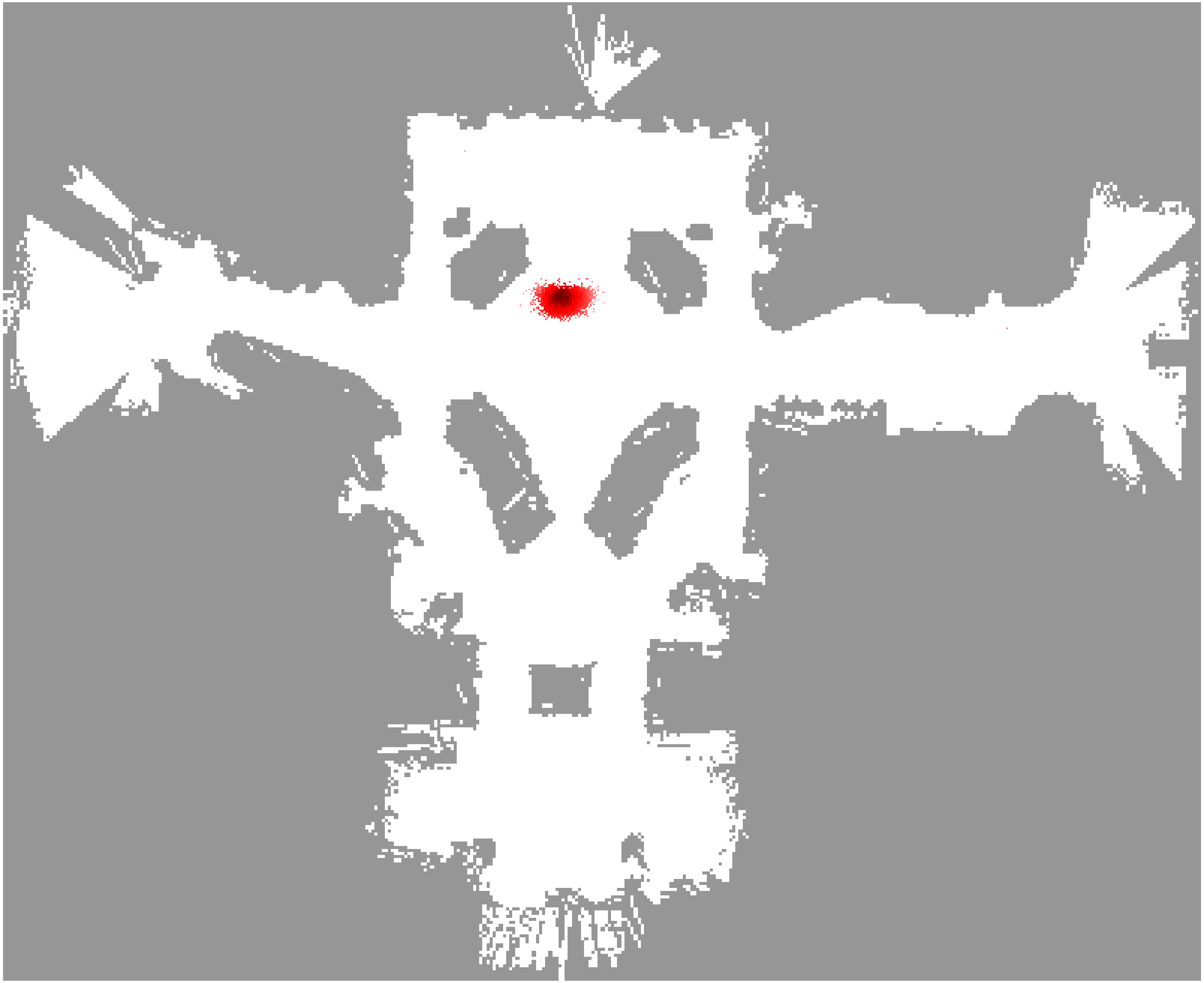


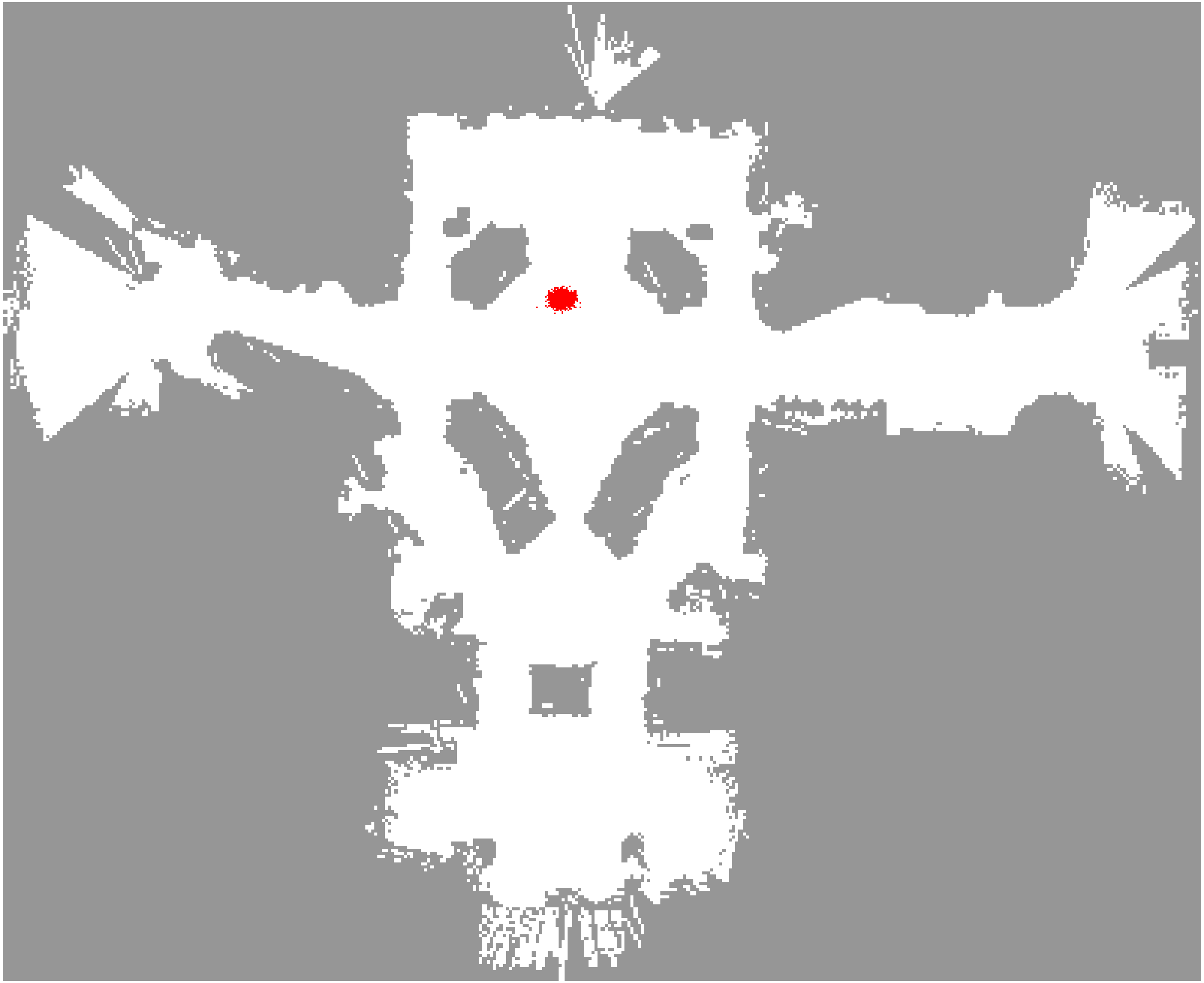


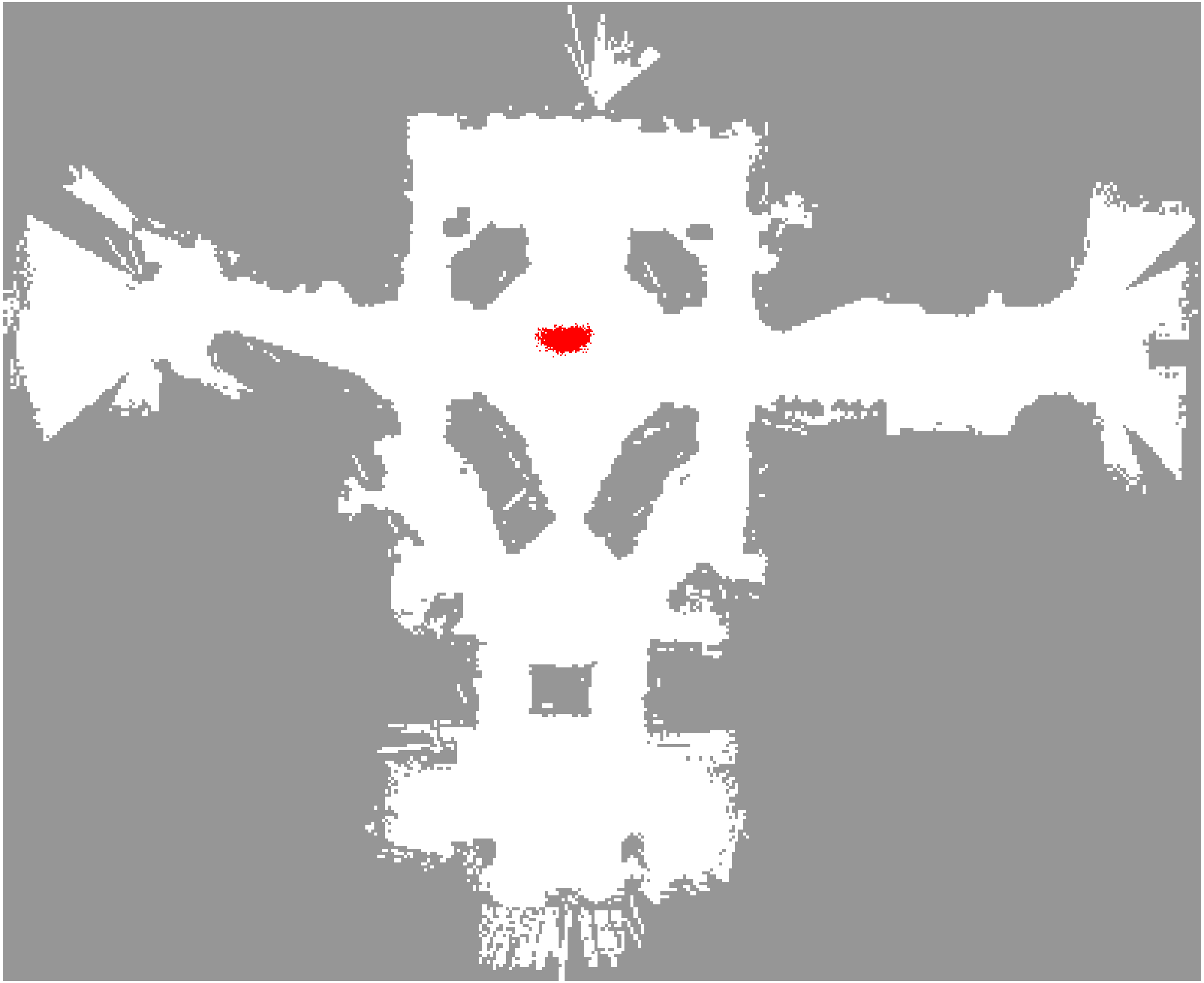


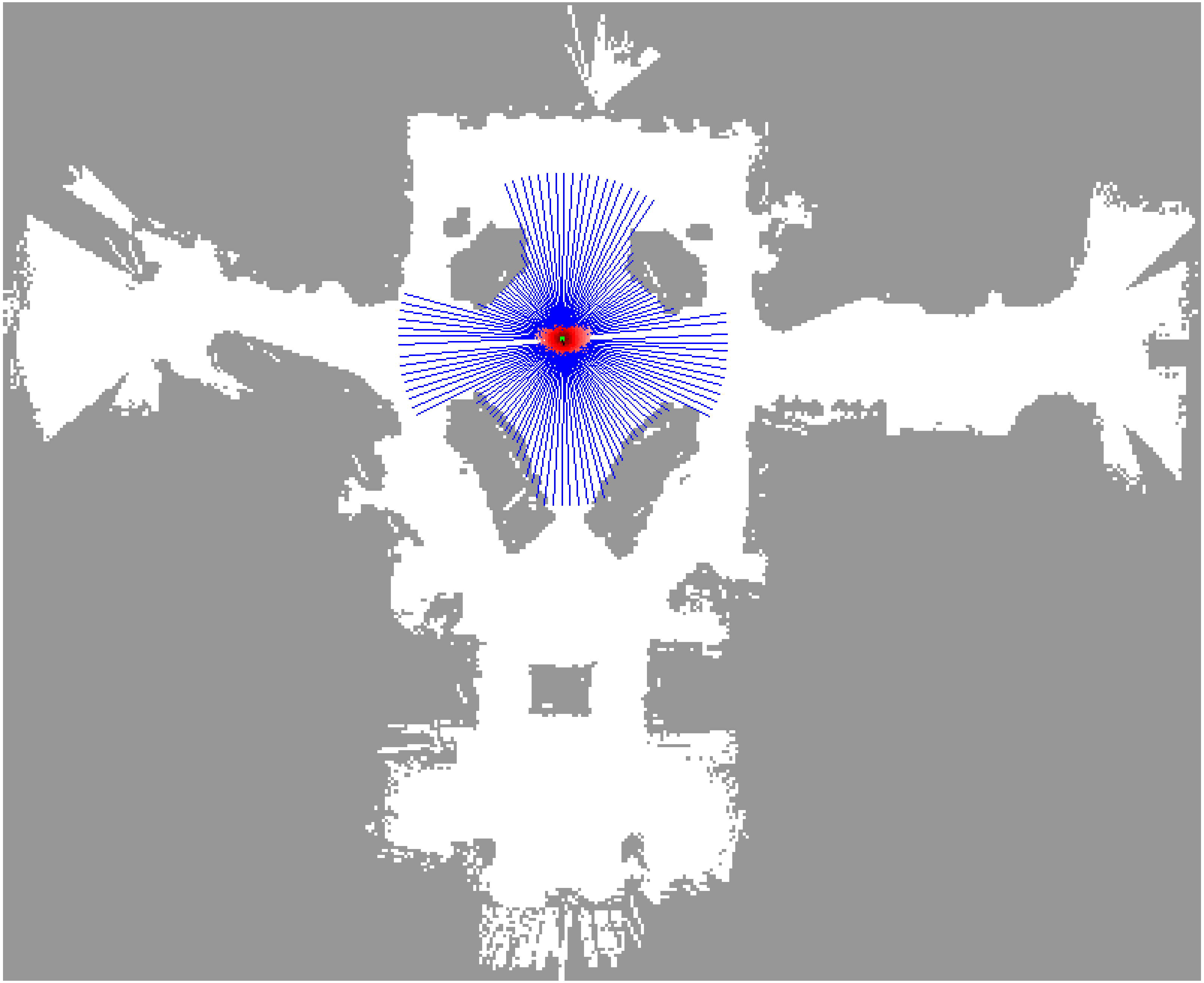




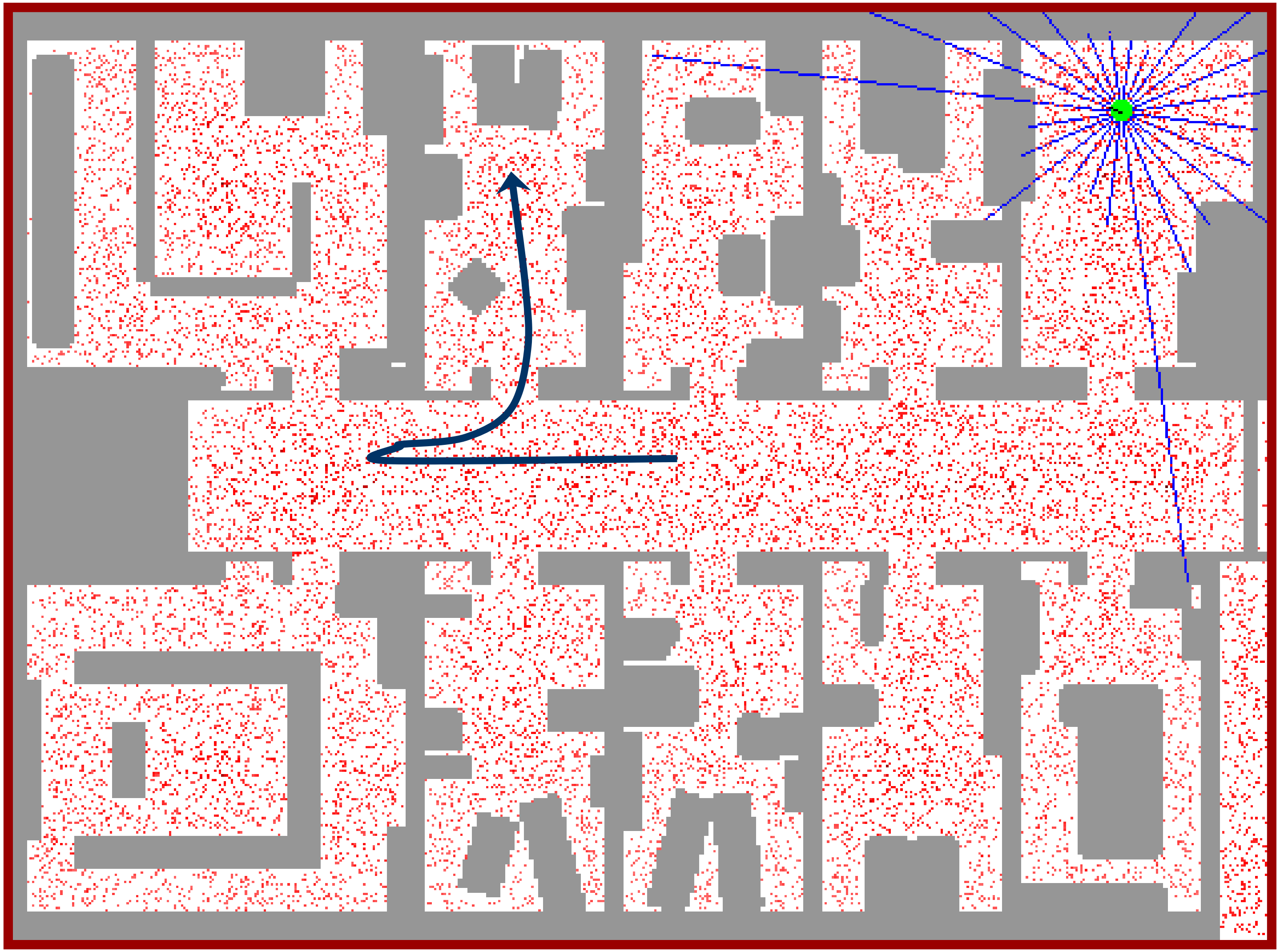




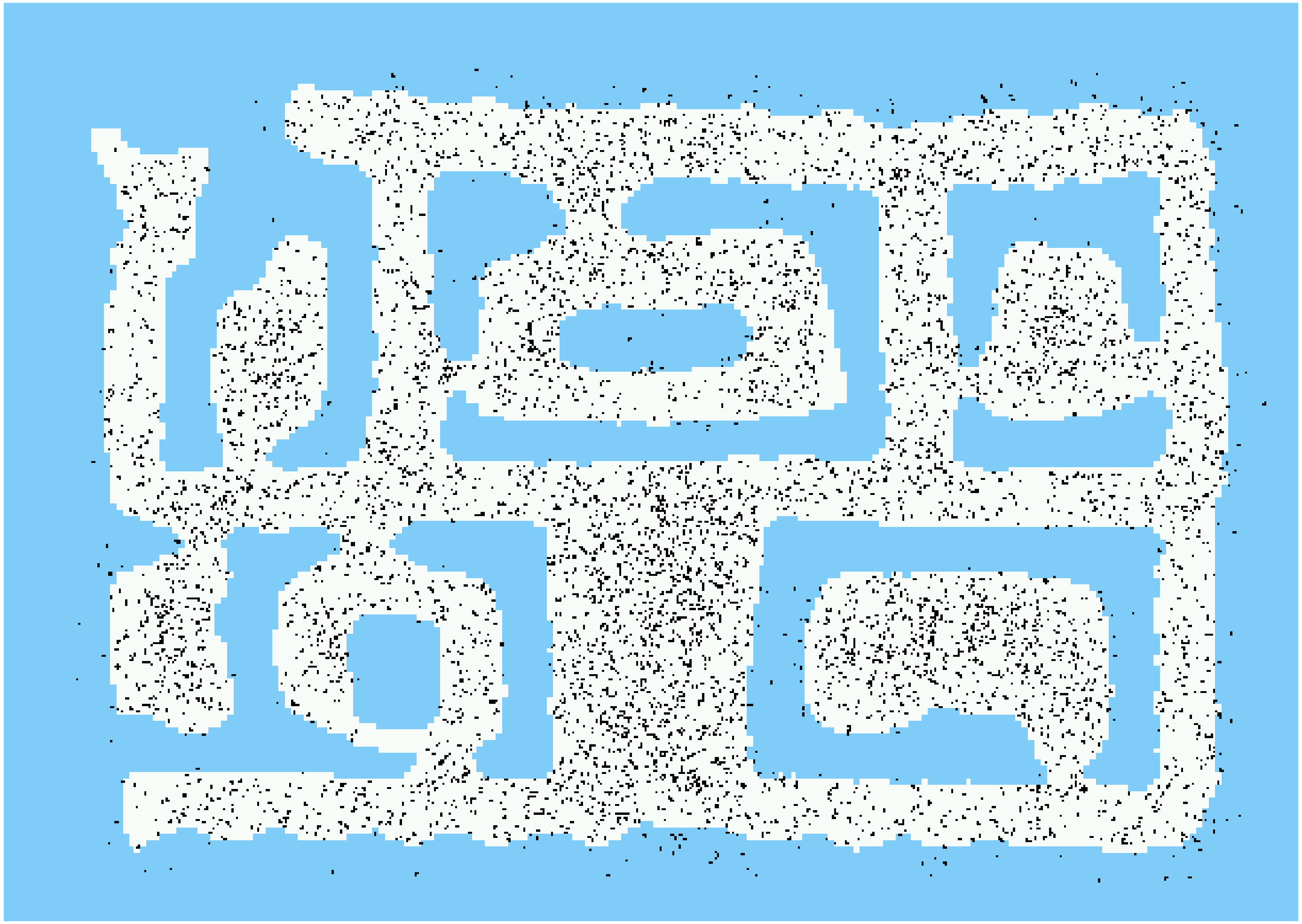




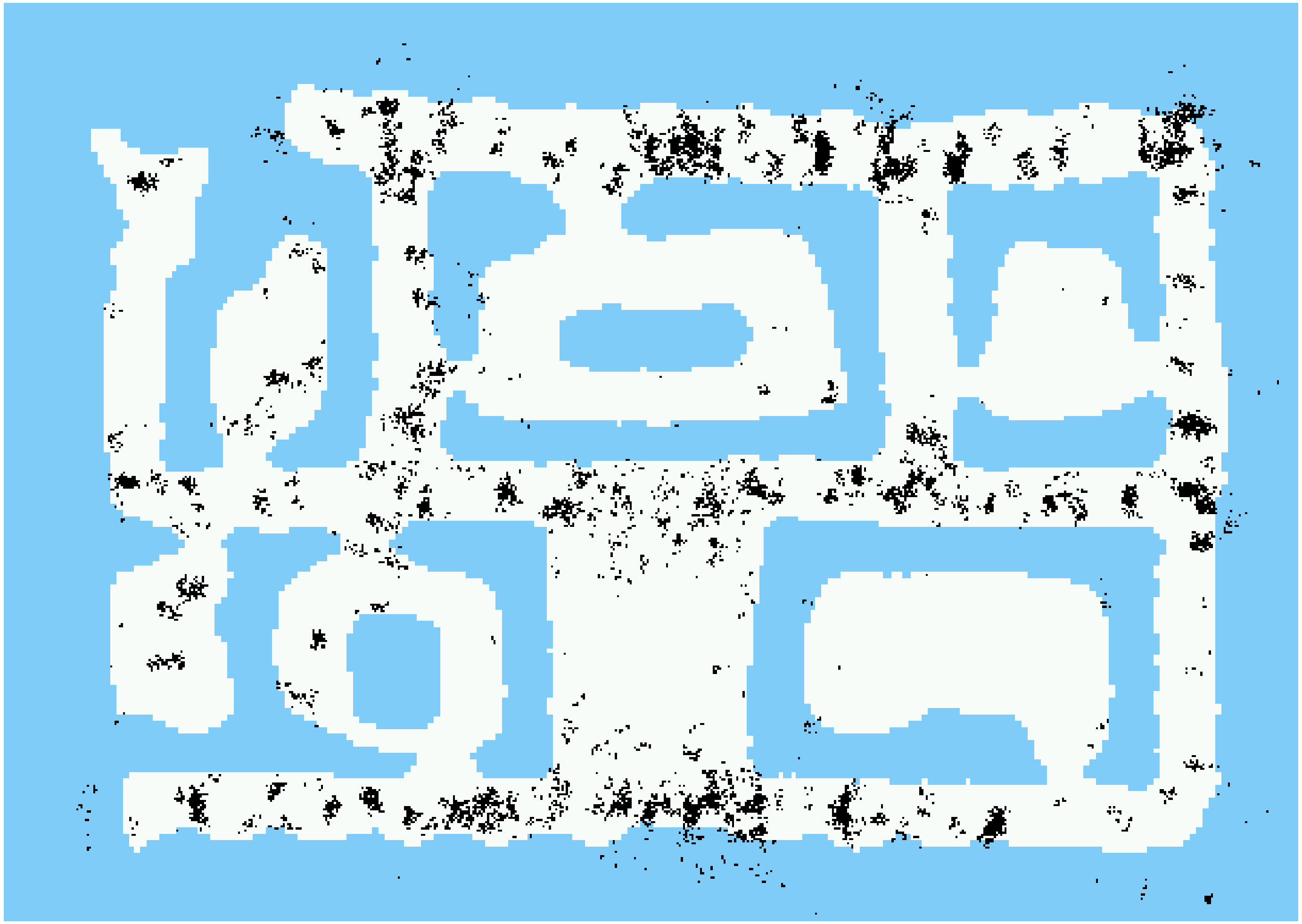
Sample-based Localization (sonar)



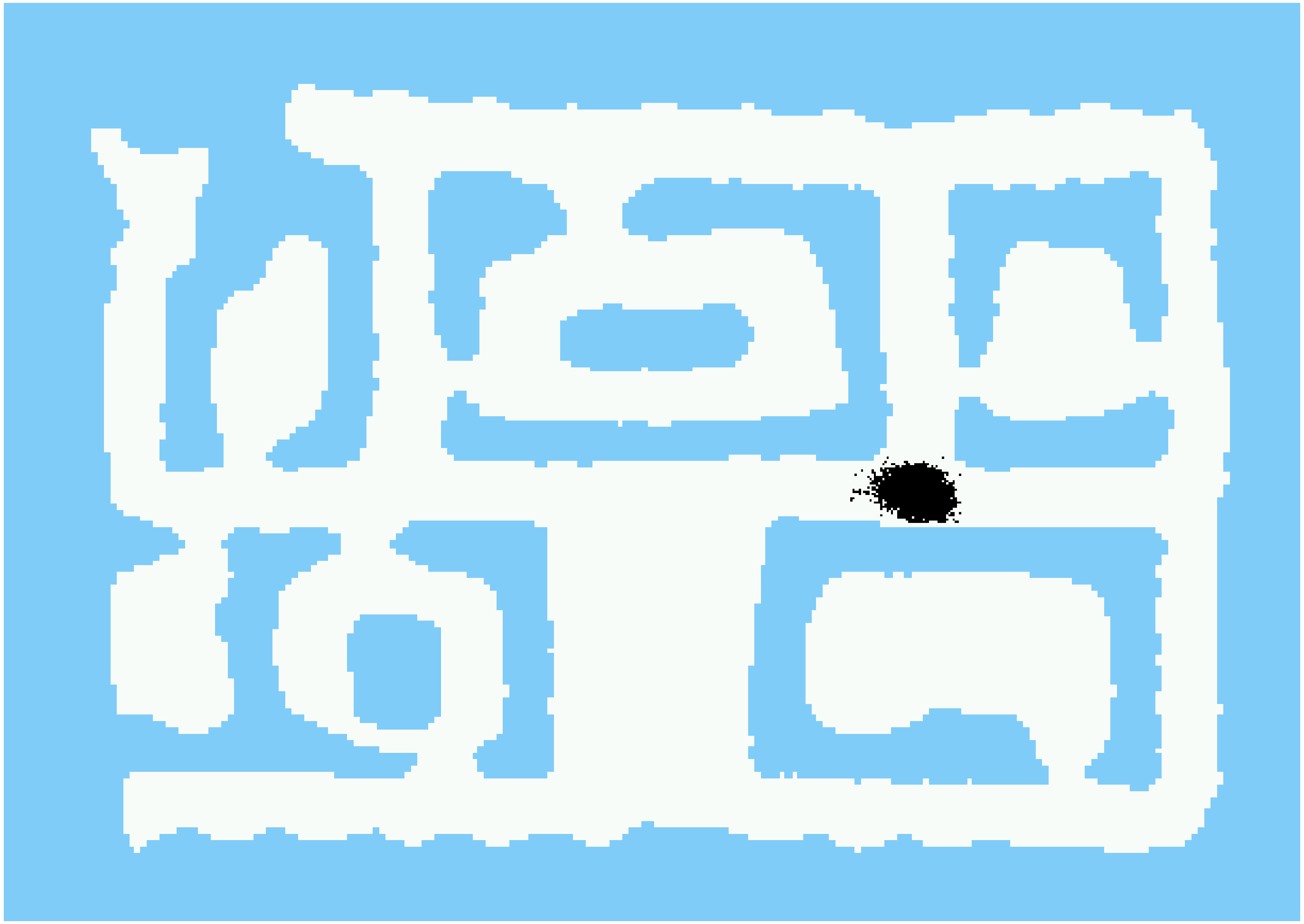
Initial Distribution



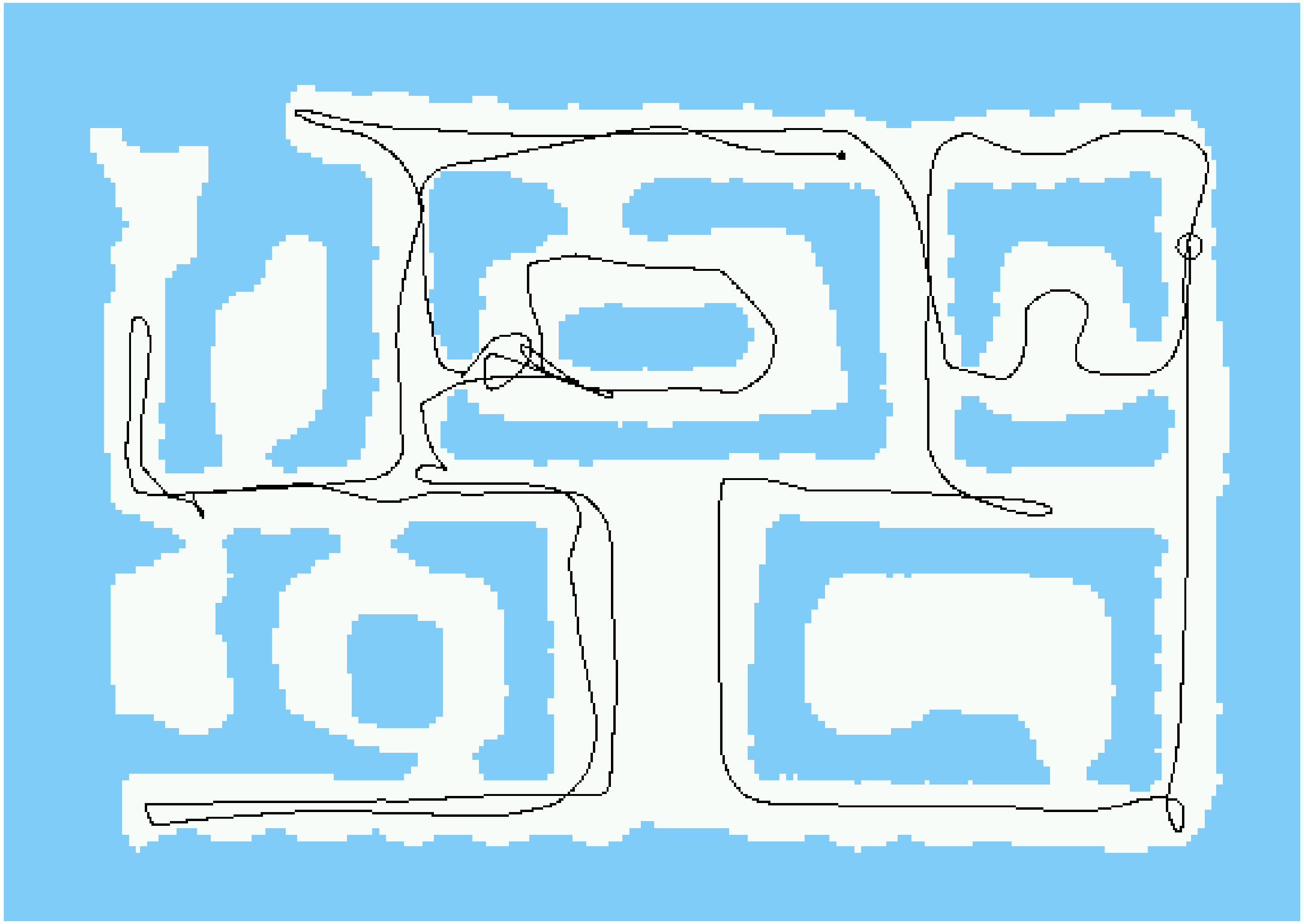
After Incorporating Ten Ultrasound Scans



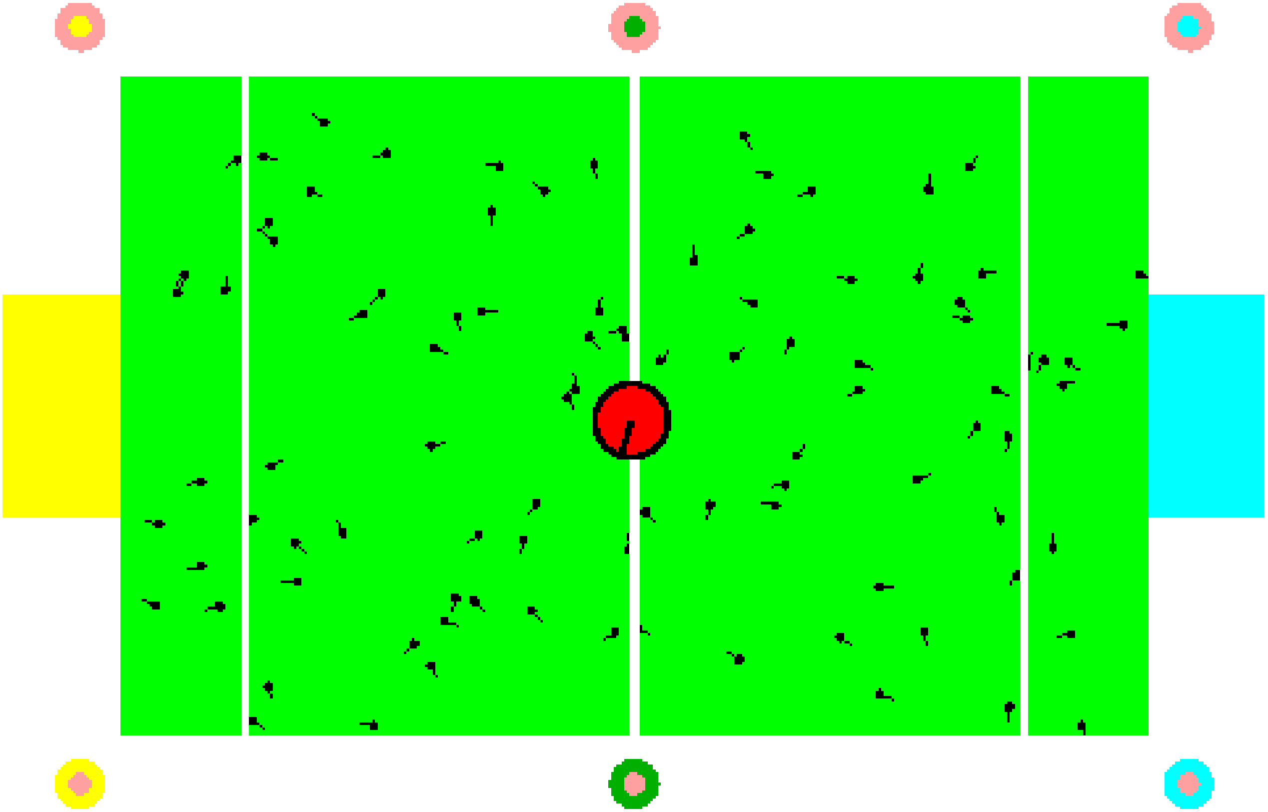
After Incorporating 65 Ultrasound Scans



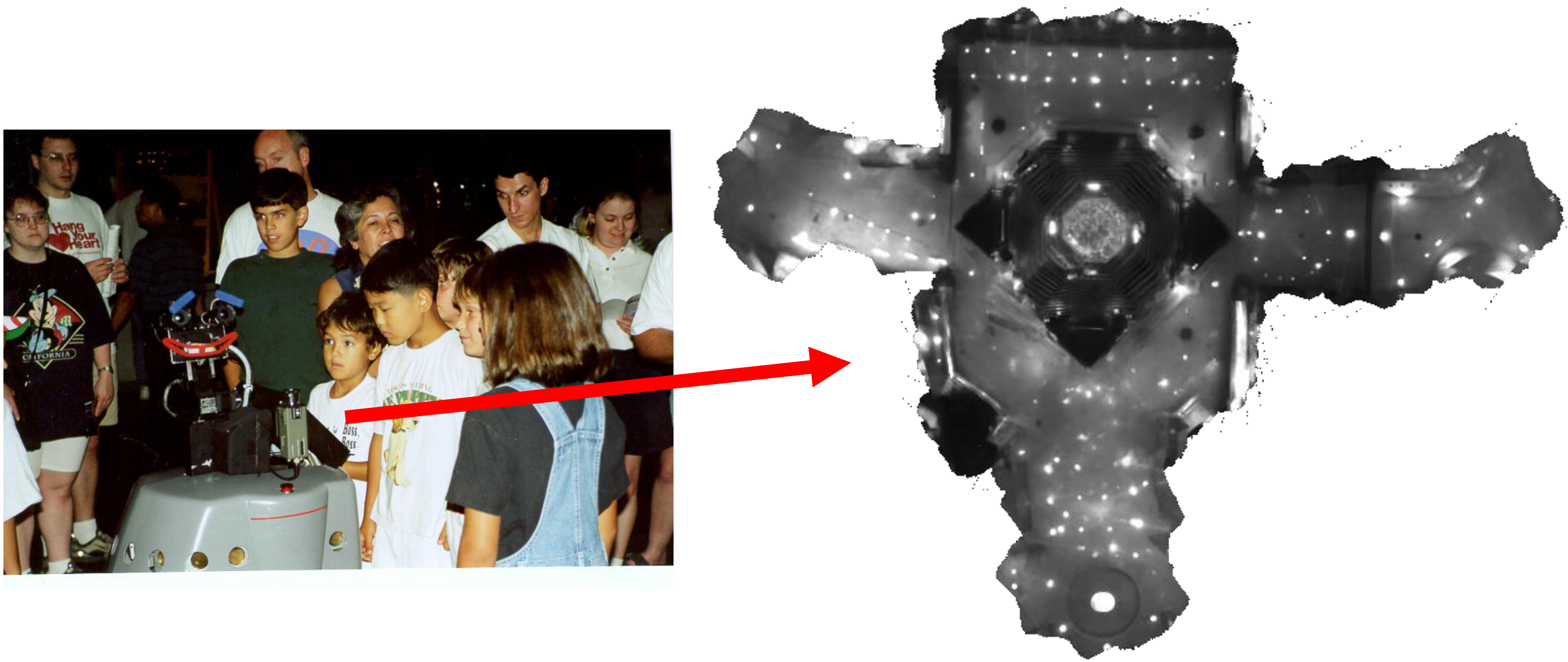
Estimated Path



Localization for AIBO robots

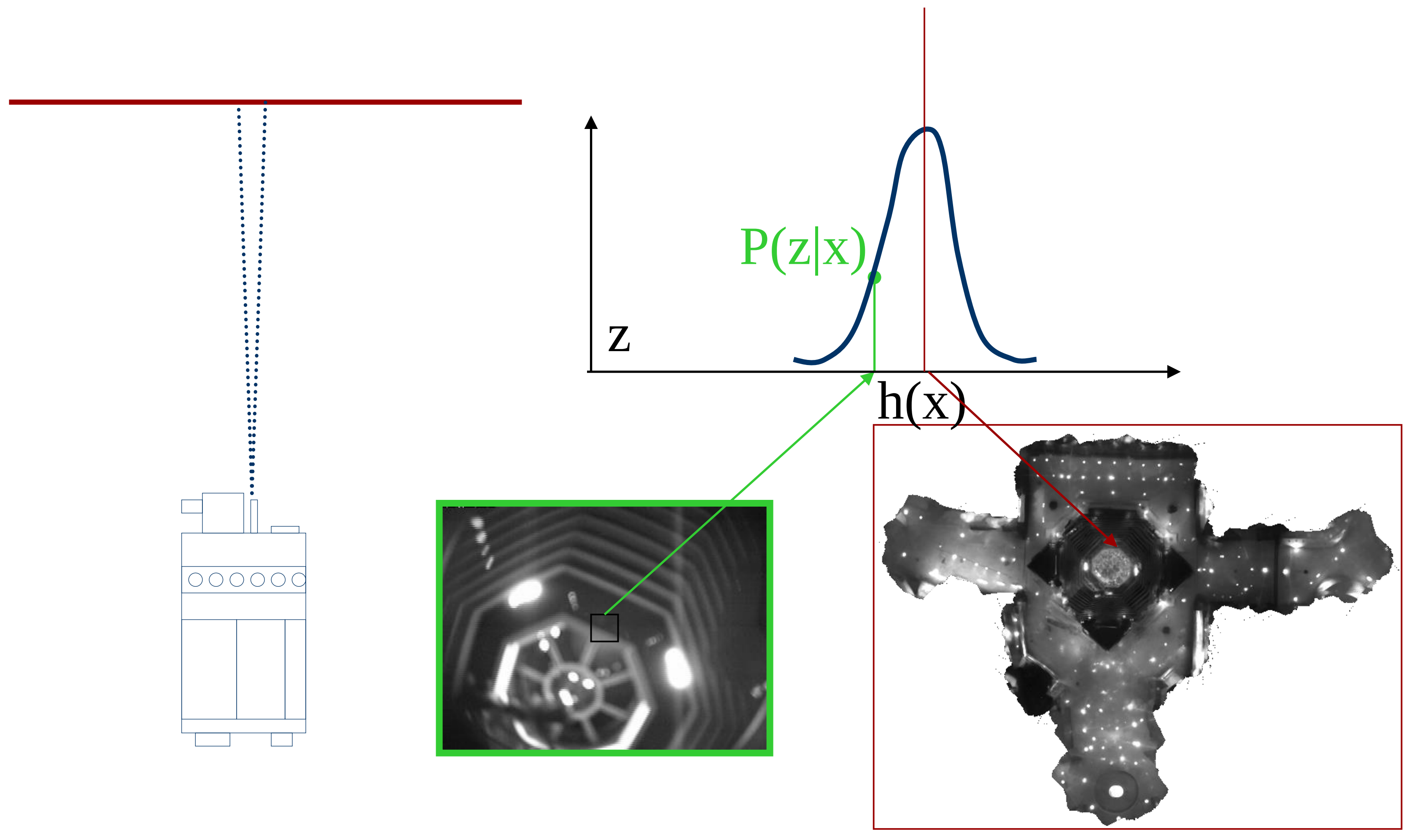


Using Ceiling Maps for Localization



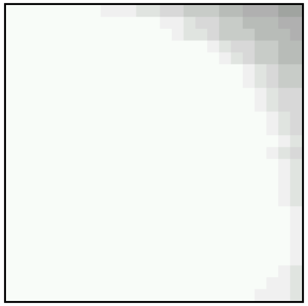
[Dellaert et al. 99]

Vision-based Localization

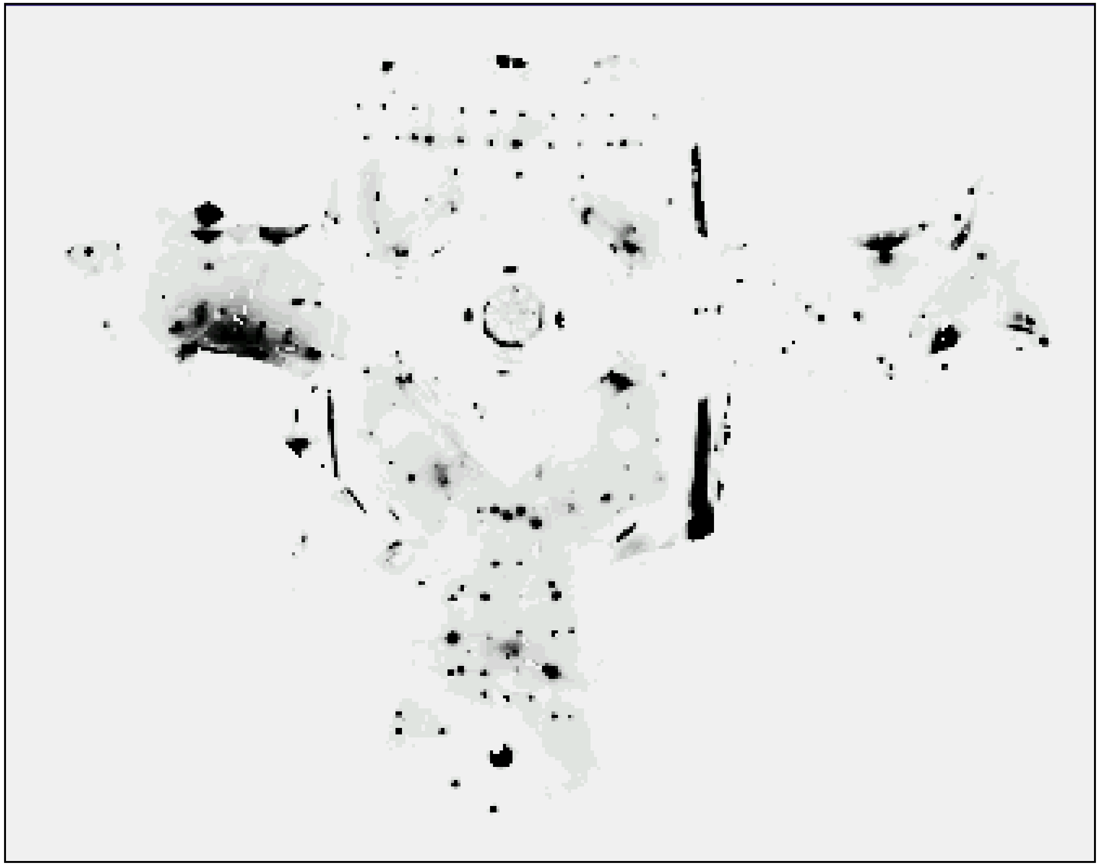


Under a Light

Measurement z:

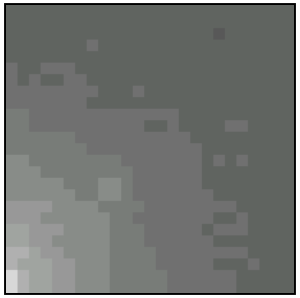


***P(z|x)*:**

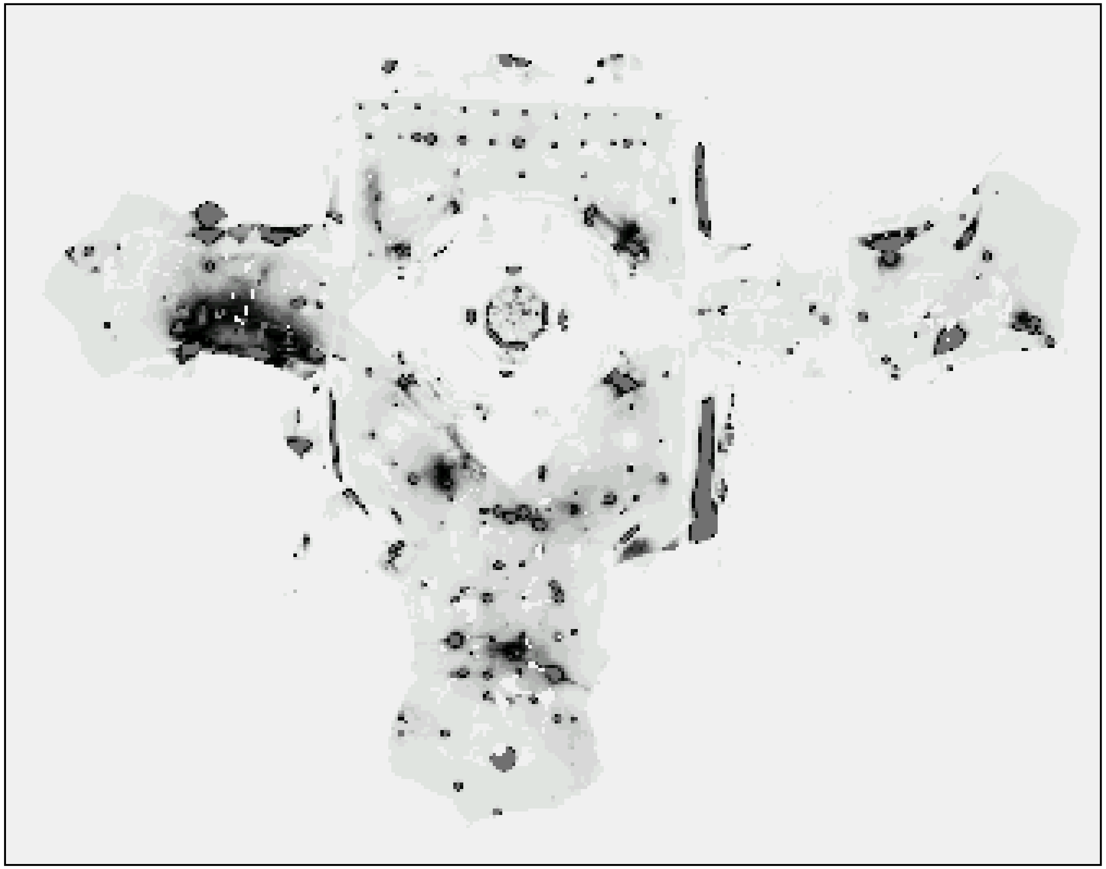


Next to a Light

Measurement z:

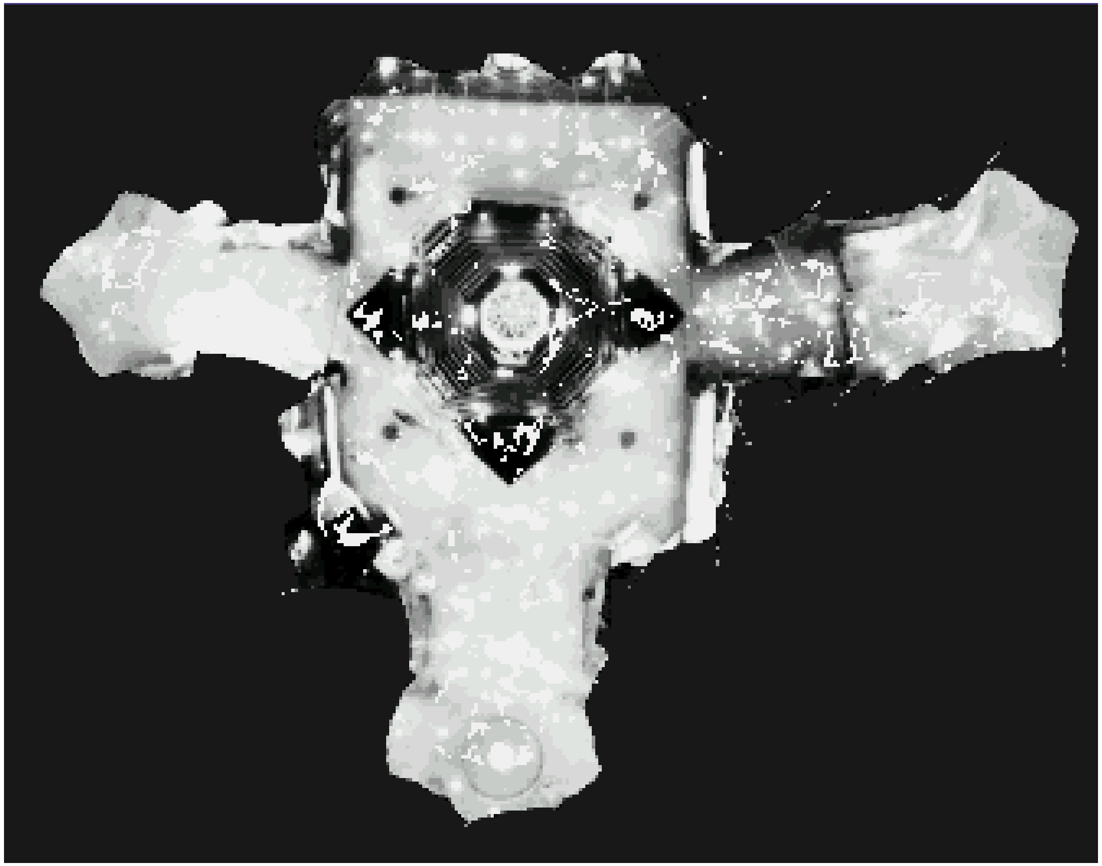


***P(z|x)*:**

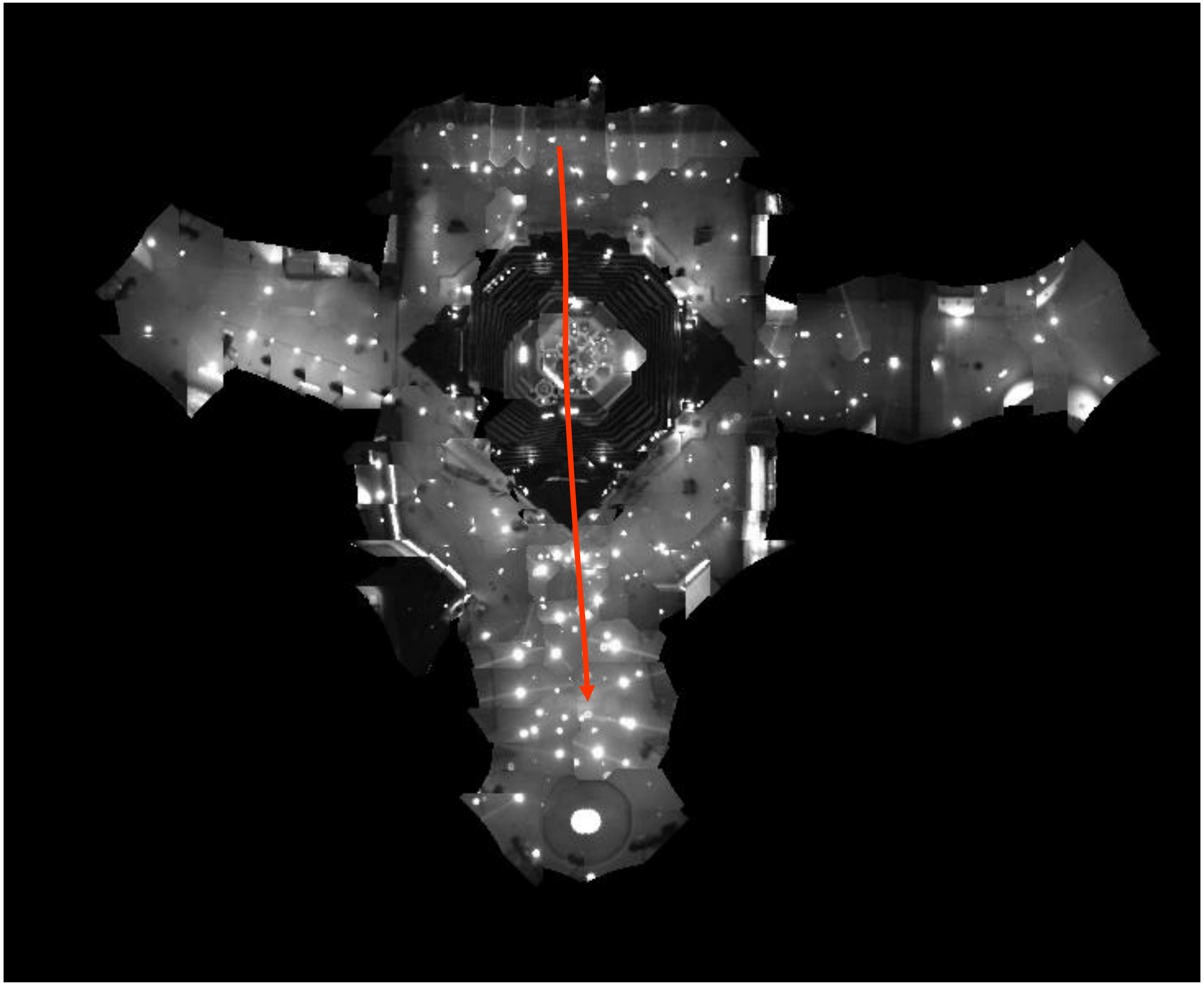


Elsewhere

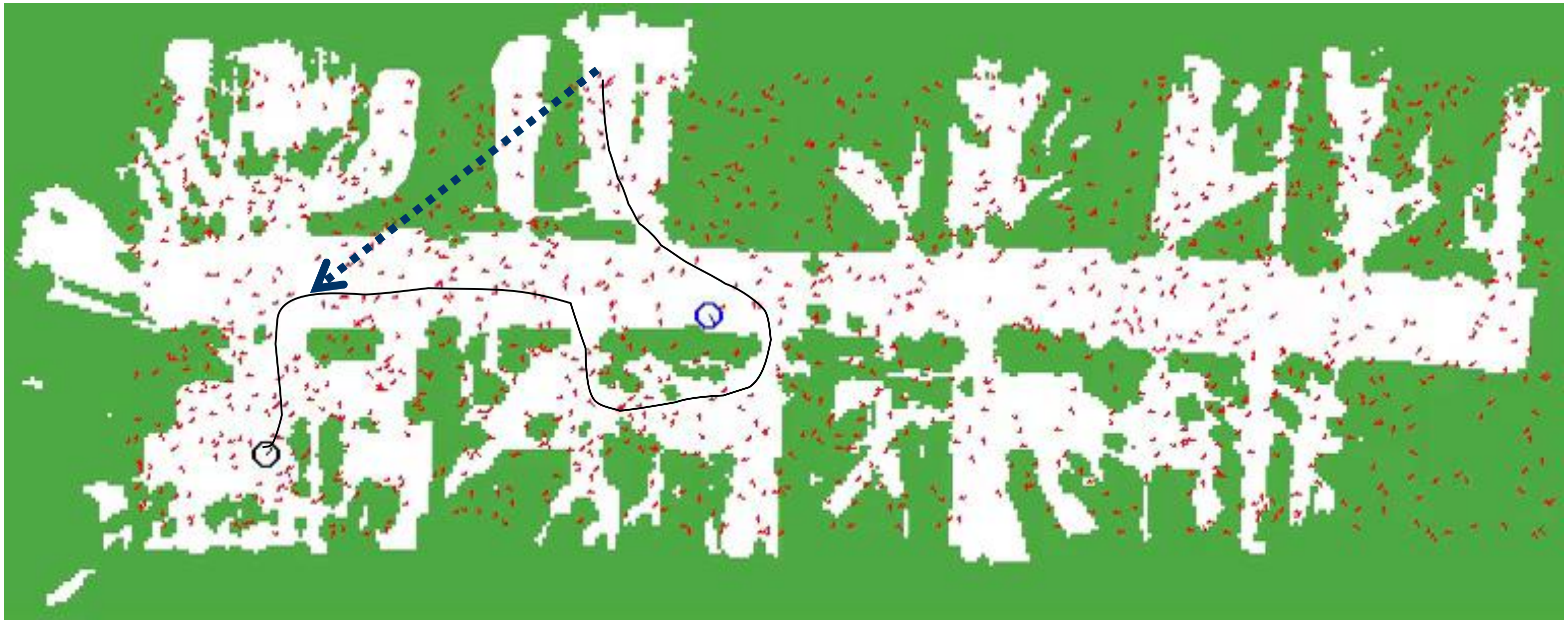
Measurement z: 

***P(z|x)*:** 

Global Localization Using Vision



Vision-based Localization



Limitations

- The approach described so far is able
  - to track the pose of a mobile robot and
  - to globally localize the robot
- How can we deal with localization errors (i.e., the kidnapped robot problem)?

Approaches

- Randomly insert a fixed number of samples with randomly chosen poses
- This corresponds to the assumption that the robot can be teleported at any point in time to an arbitrary location
- Alternatively, insert such samples inverse proportional to the average likelihood of the observations (the lower this likelihood the higher the probability that the current estimate is wrong).

Summary – Particle Filters

- Particle filters are an implementation of recursive Bayesian filtering
- They represent the posterior by a set of weighted samples
- They can model arbitrary and thus also non-Gaussian distributions
- Proposal to draw new samples
- Weights are computed to account for the difference between the proposal and the target
- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter

Summary – PF Localization

- In the context of localization, the particles are propagated according to the motion model.
- They are then weighted according to the likelihood model (likelihood of the observations).
- In a re-sampling step, new particles are drawn with a probability proportional to the likelihood of the observation.
- This leads to one of the most popular approaches to mobile robot localization