Introduction to Mobile Robotics

Path and Motion Planning

Wolfram Burgard, Michael Ruhnke, Bastian Steder

Motion Planning

Latombe (1991):

"… eminently necessary since, by definition, a robot accomplishes tasks by moving in the real world."

Goals:

- Collision-free trajectories.
- Robot should reach the goal location as quickly as possible.

… in Dynamic Environments

- How to react to unforeseen obstacles?
  - efficiency
  - reliability

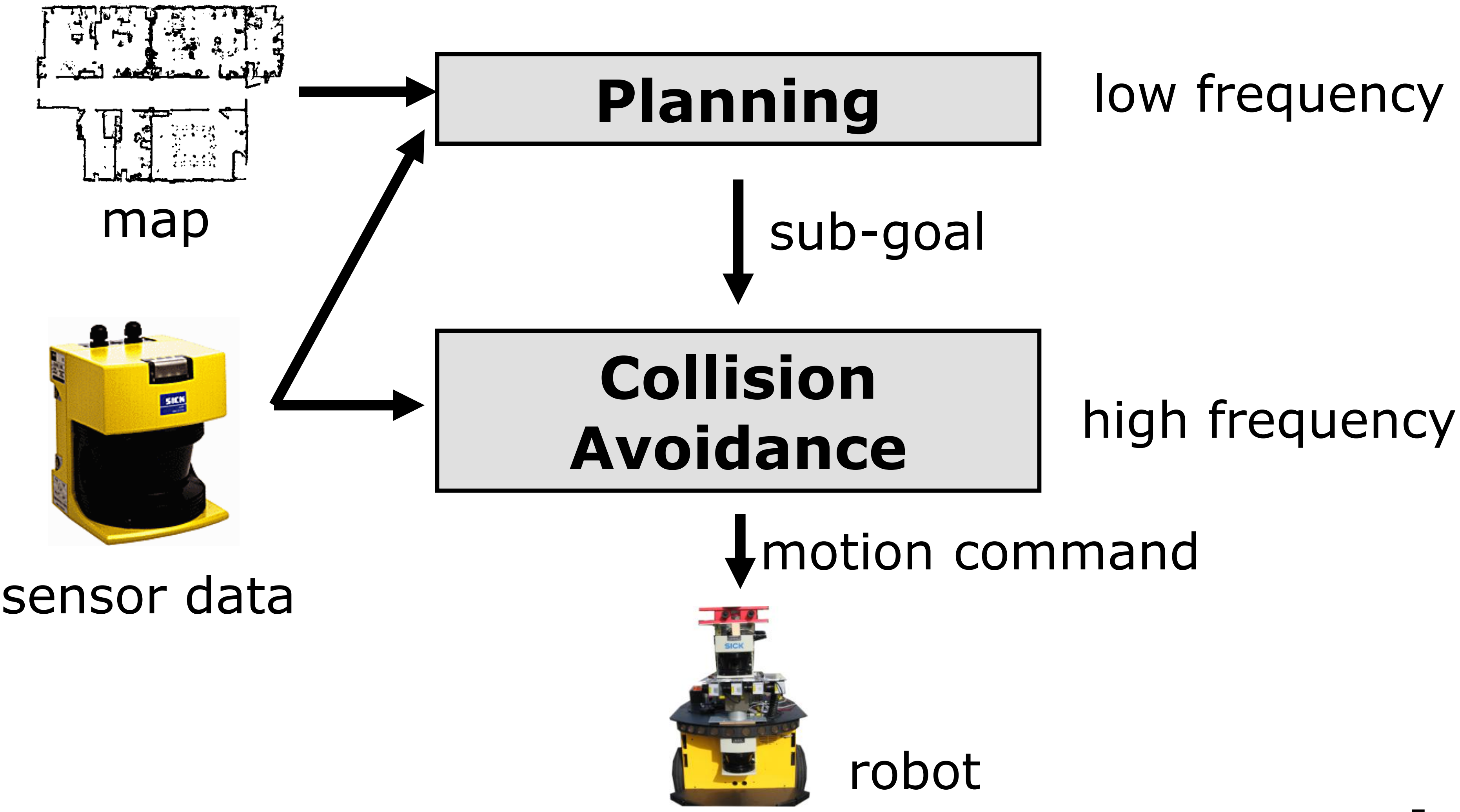
- Dynamic Window Approaches
  [Simmons, 96], [Fox et al., 97], [Brock & Khatib, 99]
- Grid-map-based planning
  [Konolige, 00]
- Nearness-Diagram-Navigation
  [Minguez at al., 2001, 2002]
- Vector-Field-Histogram+
  [Ulrich & Borenstein, 98]
- A*, D*, D* Lite, ARA*, …

Two Challenges

- Calculate the optimal path taking potential uncertainties in the actions into account

- Quickly generate actions in the case of unforeseen objects

Classic Two-layered Architecture



map

sensor data

Planning

low frequency

sub-goal

Collision Avoidance

high frequency

motion command

robot

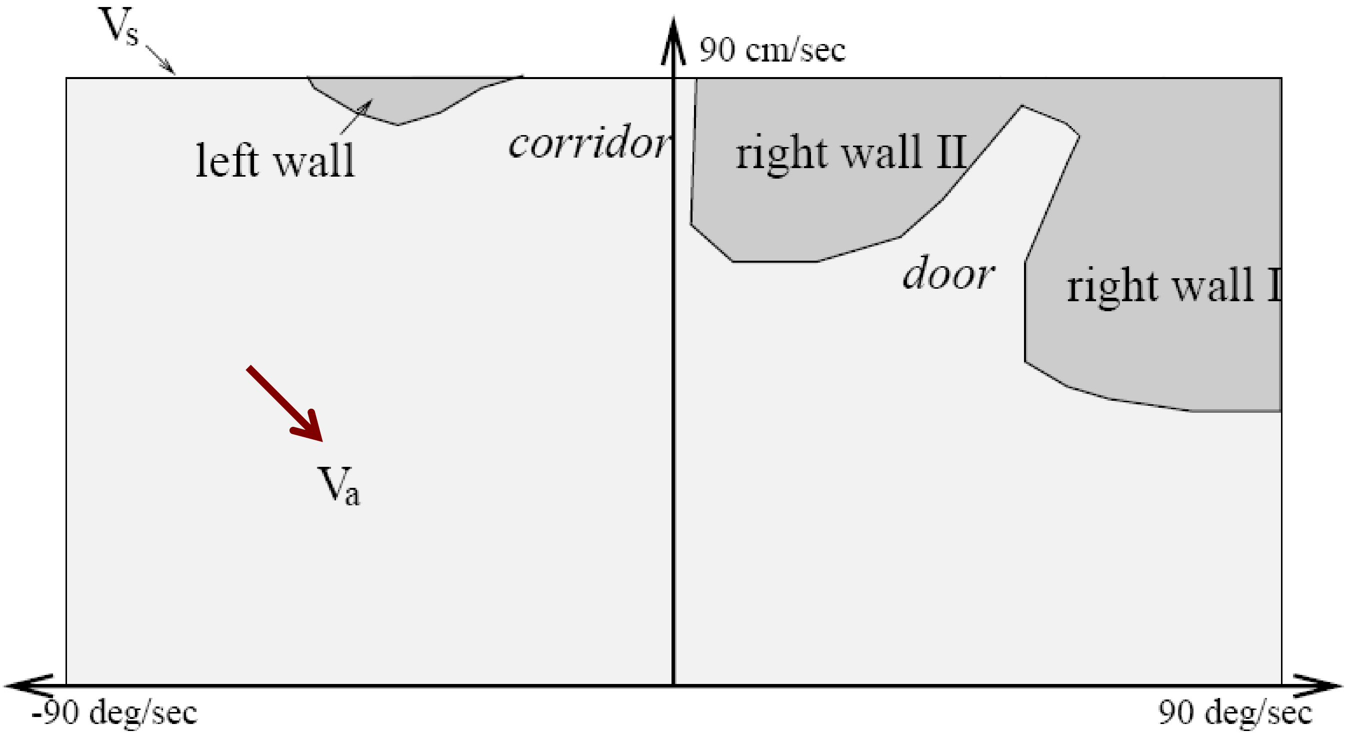
Dynamic Window Approach

- **Collision avoidance:** Determine collision-free trajectories using geometric operations
- Here: Robot moves on circular arcs
- Motion commands (v,ω)
- Which (v,ω) are admissible and reachable?

Admissible Velocities

- Speeds are admissible if the robot would be able to stop before reaching the obstacle

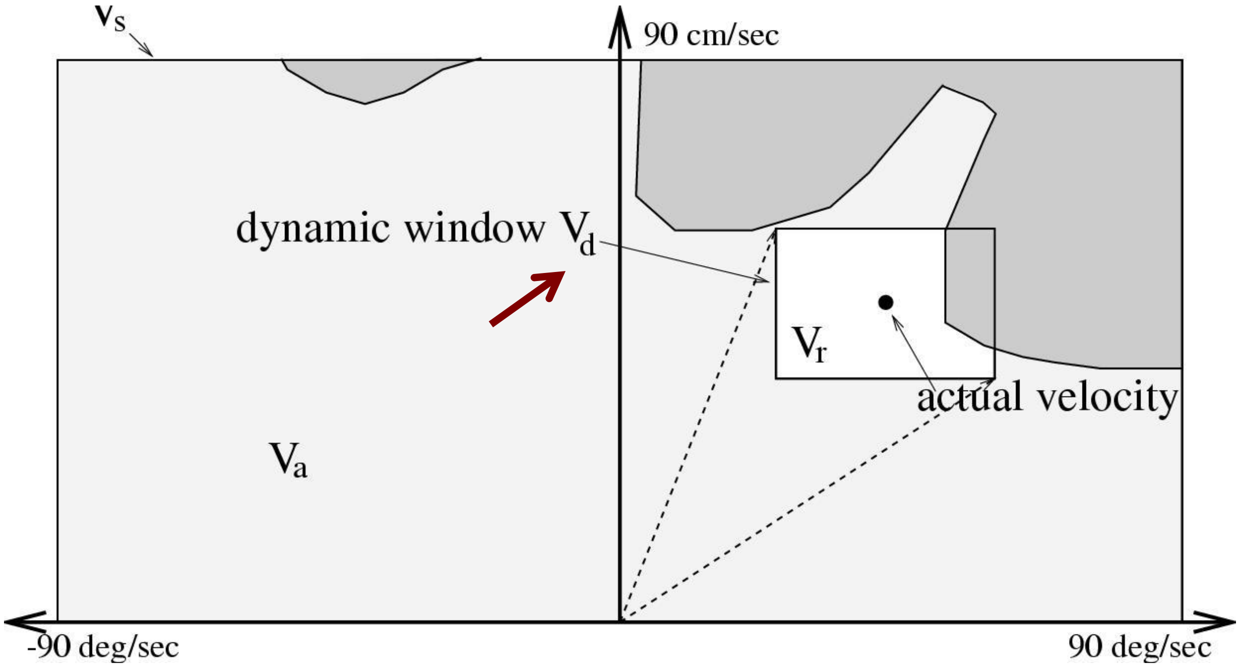
$$V_a = \{(v,\omega) \;\mid\; v \leq \sqrt{2\text{dist}(v,\omega)a_{trans}} \;\wedge\; \omega \leq \sqrt{2\text{dist}(v,\omega)a_{rot}}\}$$



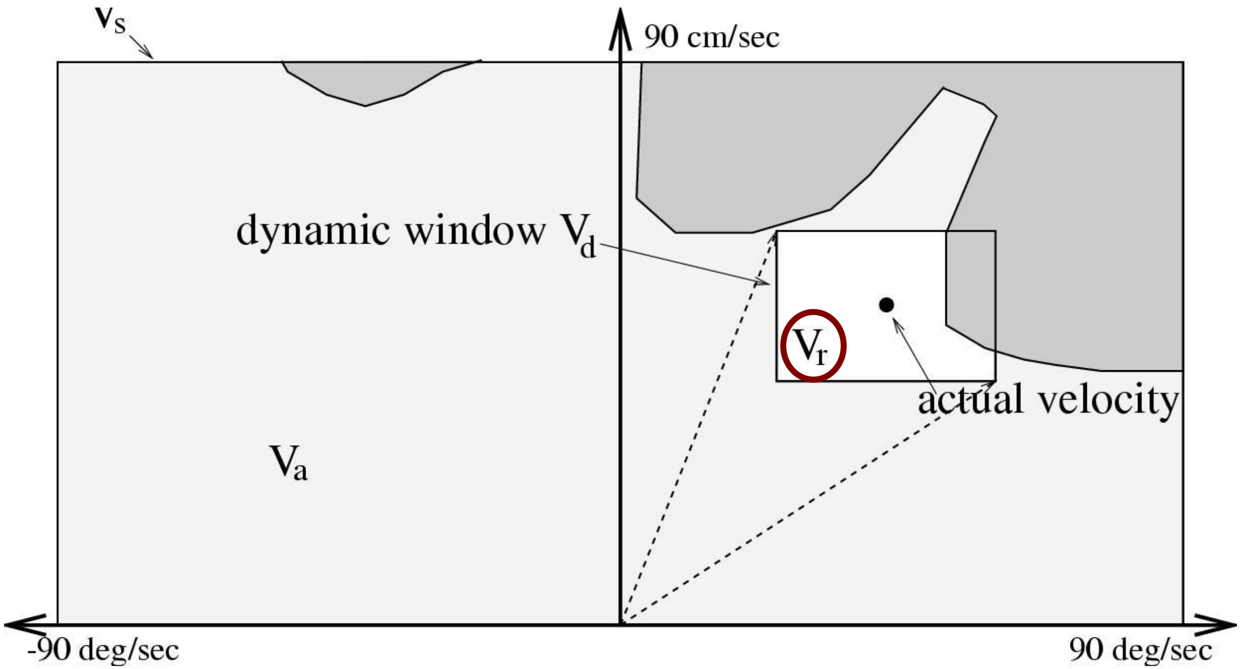
Reachable Velocities

- Speeds that are reachable by acceleration

$$V_d = \{(v,\omega) \;\mid\; v \in [v - a_{trans}t, v + a_{trans}t] \;\wedge\; \omega \in [\omega - a_{rot}t, \omega + a_{rot}t]\}$$



DWA Search Space



- V_s = all possible speeds of the robot.
- V_a = obstacle free area.
- V_d = speeds reachable within a certain time frame based on possible accelerations.

$$V_r = V_s \cap V_a \cap V_d$$

Dynamic Window Approach

- How to choose <v,ω>?
- Steering commands are chosen by a heuristic navigation function.
- This function tries to minimize the travel-time by "**driving fast** into the **right direction**."

Dynamic Window Approach

- Heuristic navigation function.
- Planning restricted to <x,y>-space.
- No planning in the velocity space.

Navigation Function: [Brock & Khatib, 99]

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

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Maximizes velocity.

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Maximizes velocity.

Considers cost to reach the goal.

Dynamic Window Approach

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Maximizes velocity.

Considers cost to reach the goal.

Follows grid based path computed by A*.

Dynamic Window Approach

- Heuristic navigation function.
- Planning restricted to <x,y>-space.
- No planning in the velocity space.

Navigation Function:

Goal nearness

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Maximizes velocity

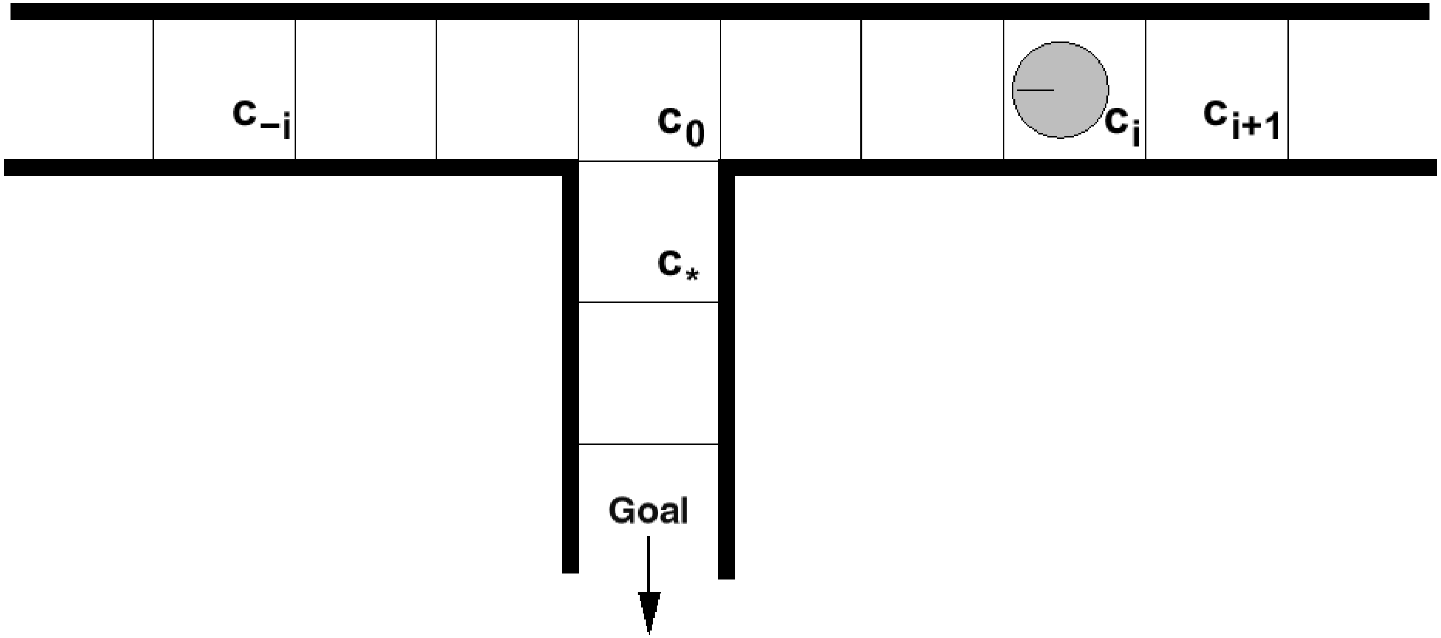
Considers cost to reach the goal

Follows grid based path computed by A*

Dynamic Window Approach

- Reacts quickly.
- Low computational requirements.
- Guides a robot along a collision-free path.
- Successfully used in a lot of real-world scenarios.
- Resulting trajectories sometimes sub-optimal.
- Local minima might prevent the robot from reaching the goal location.

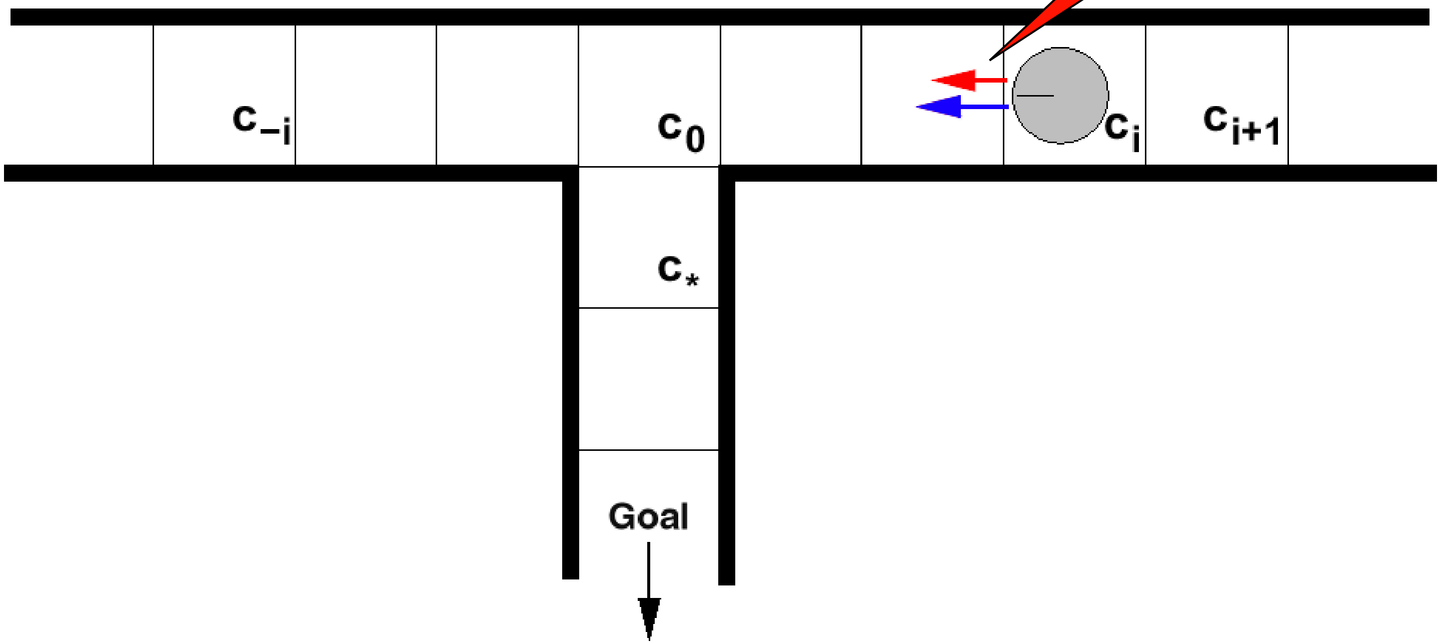
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs

Robot’s velocity.



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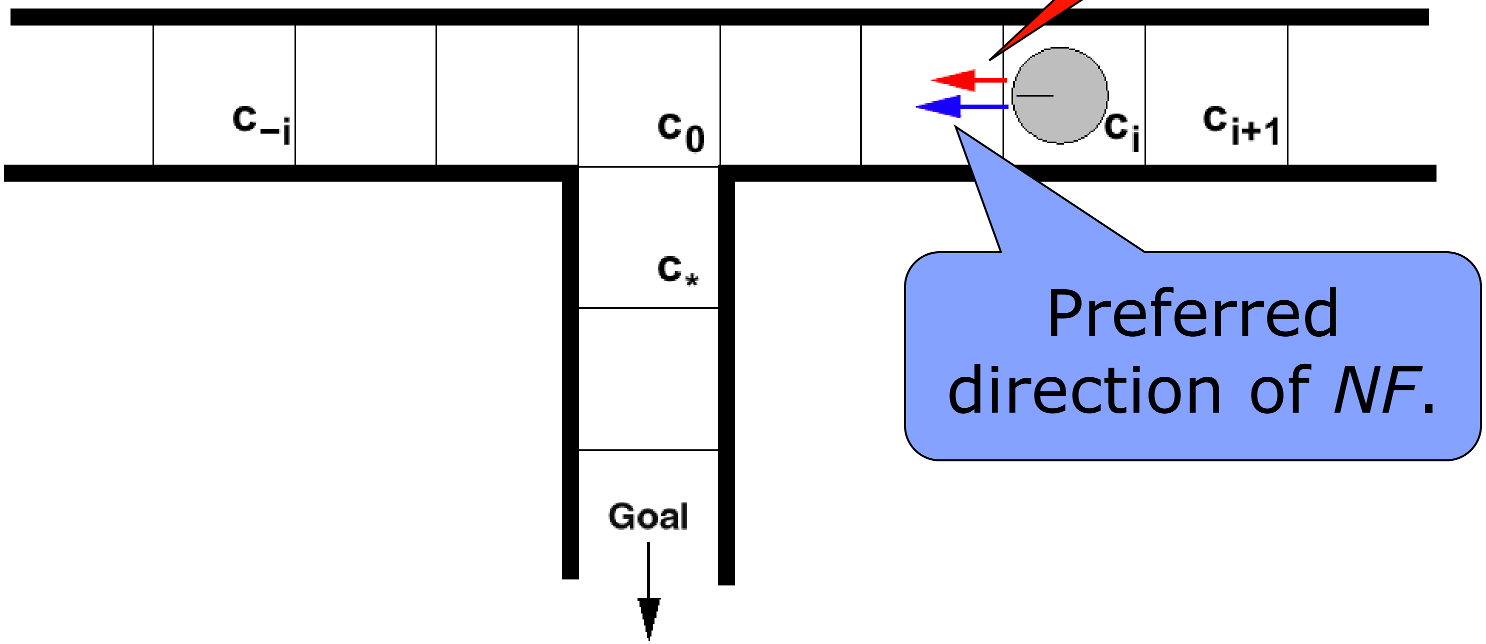
Problems of DWAs

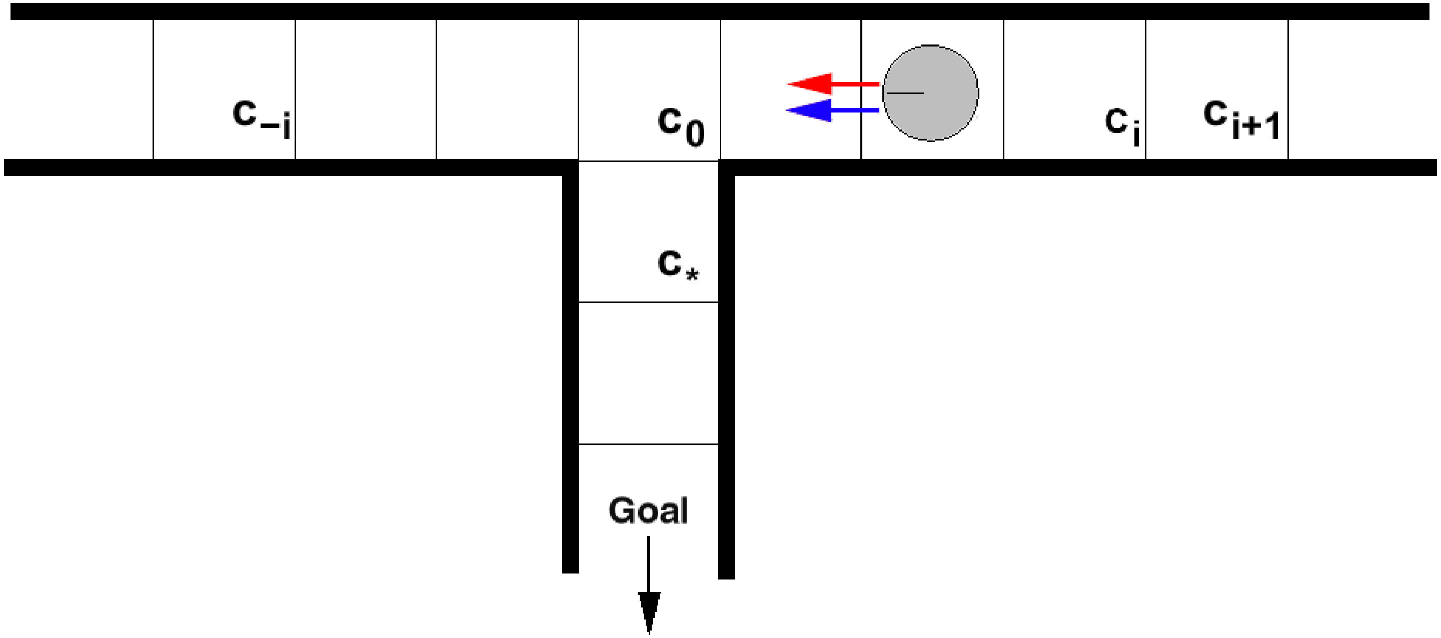
Robot's velocity.



Preferred direction of *NF*.

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



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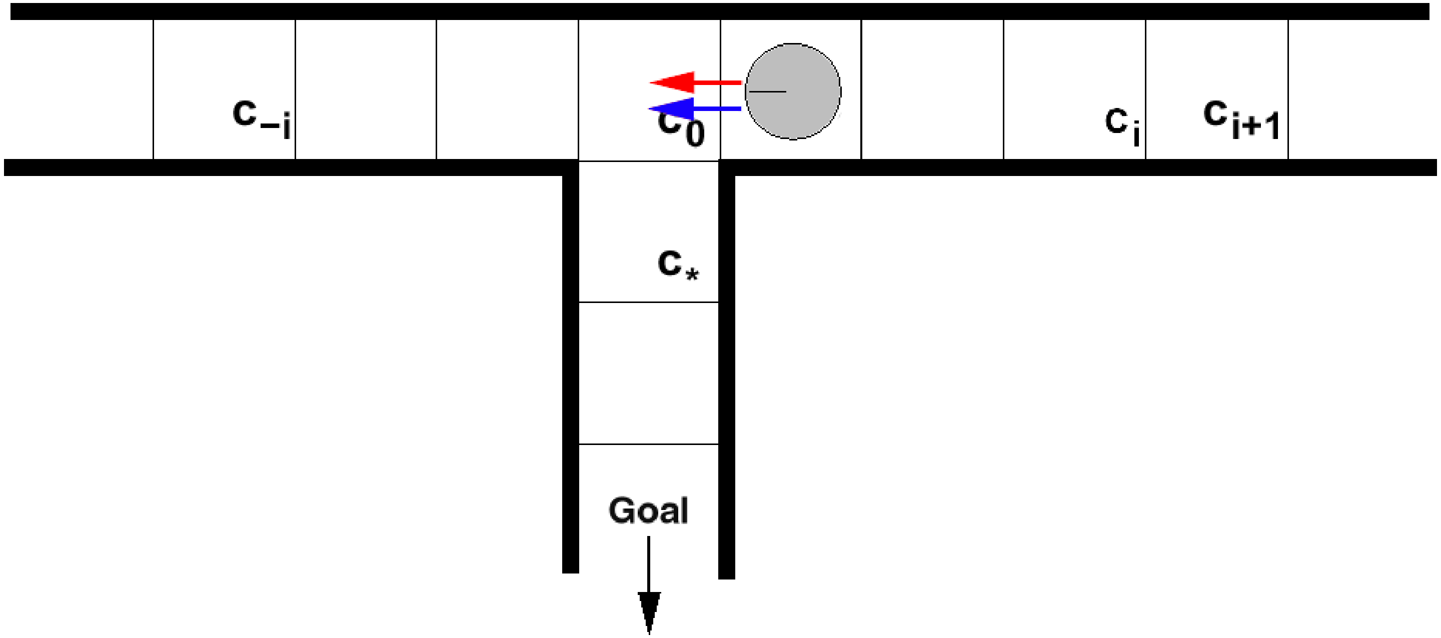
Problems of DWAs



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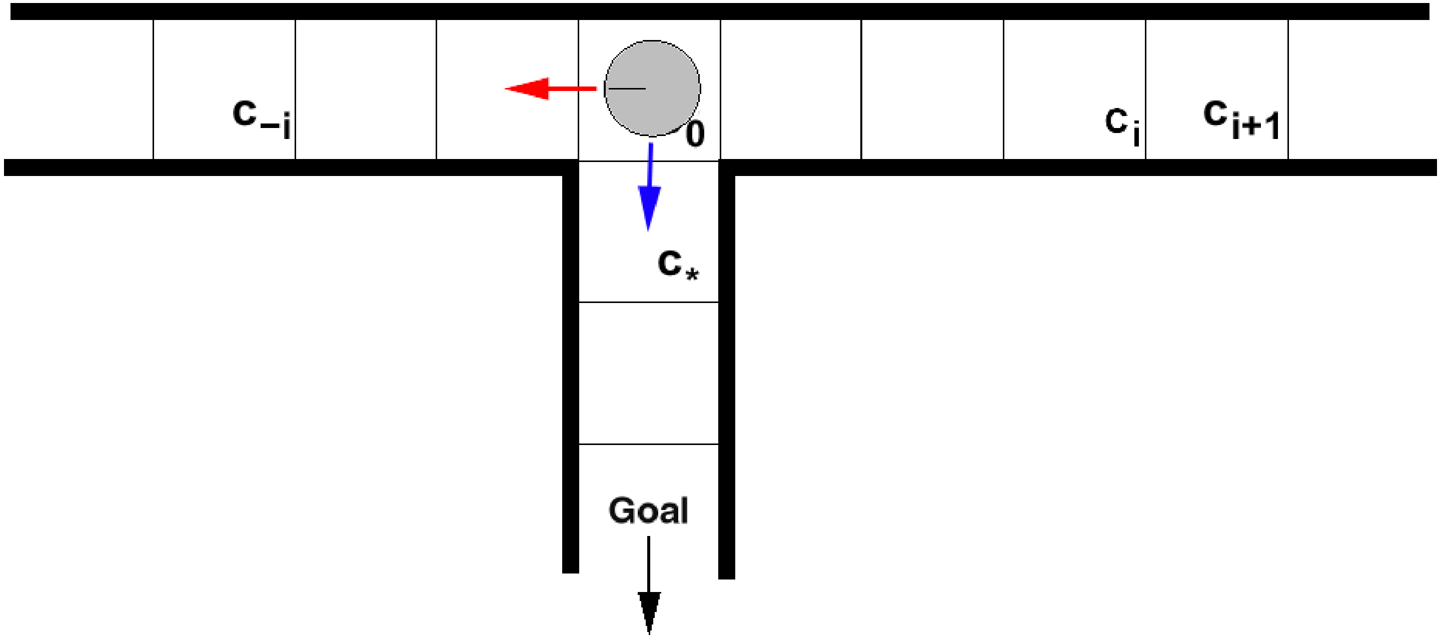
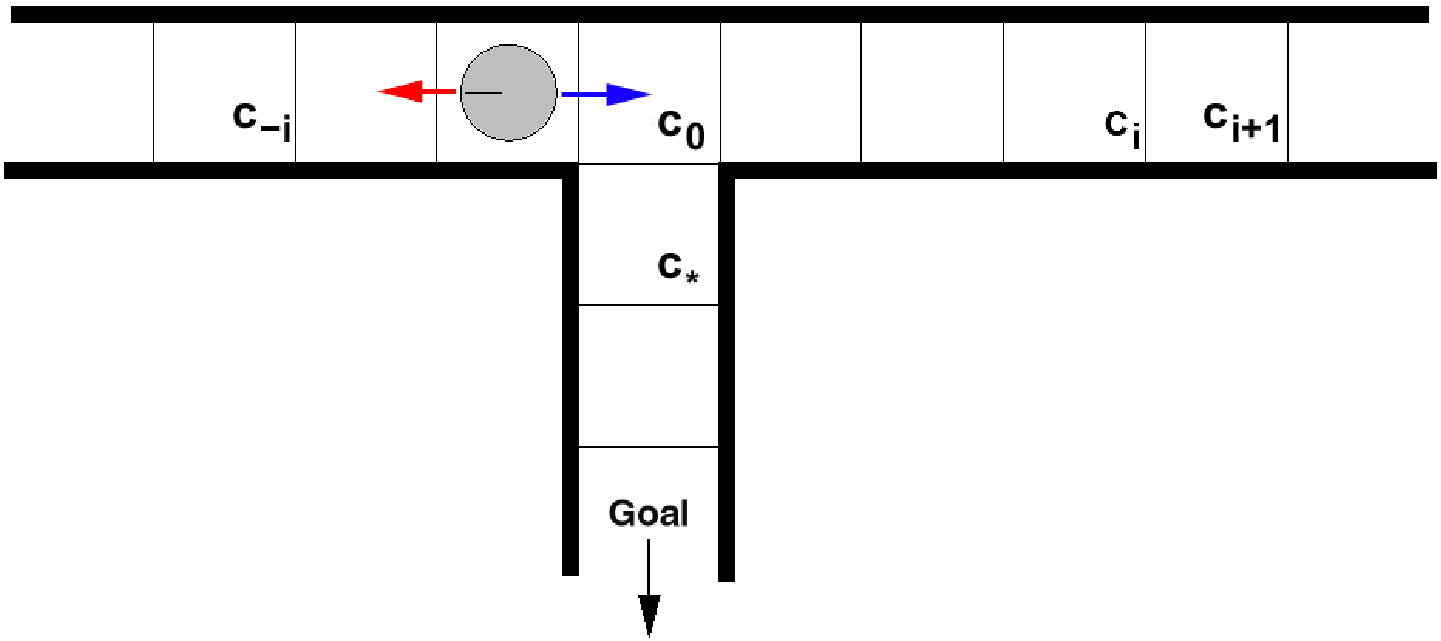
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

The robot drives too fast at c_0 to enter corridor facing south.

Problems of DWAs



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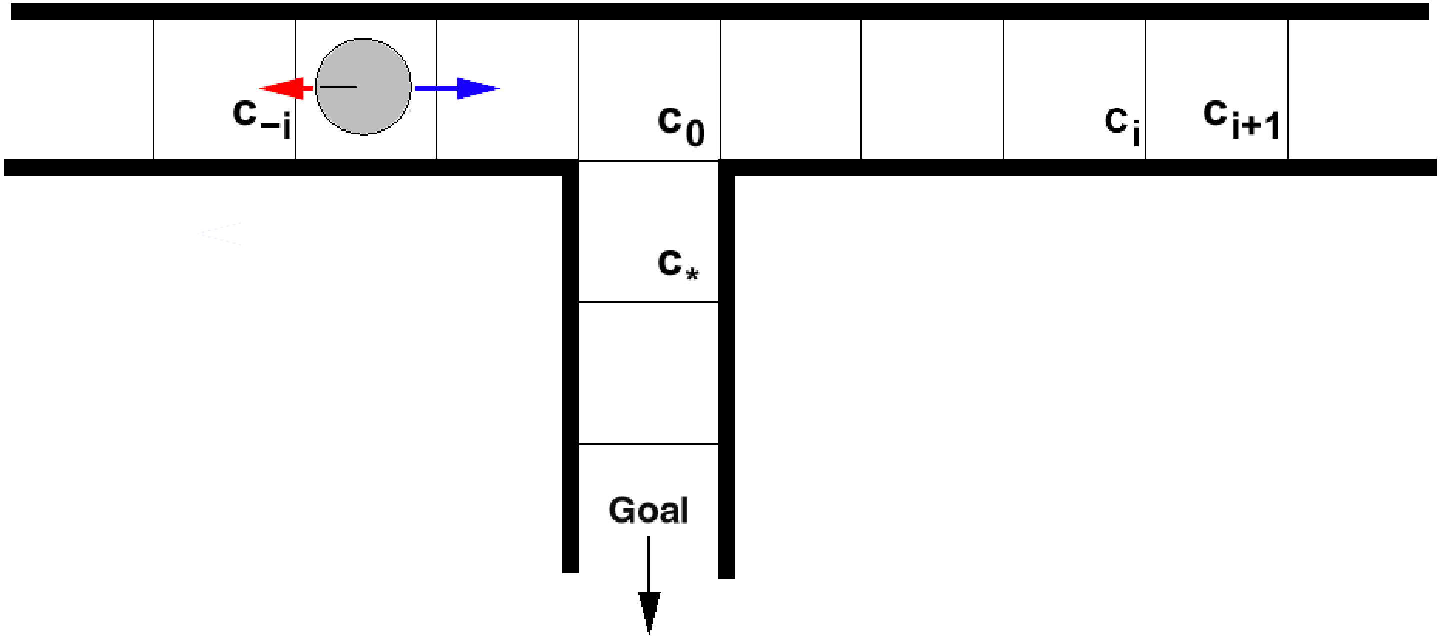
Problems of DWAs



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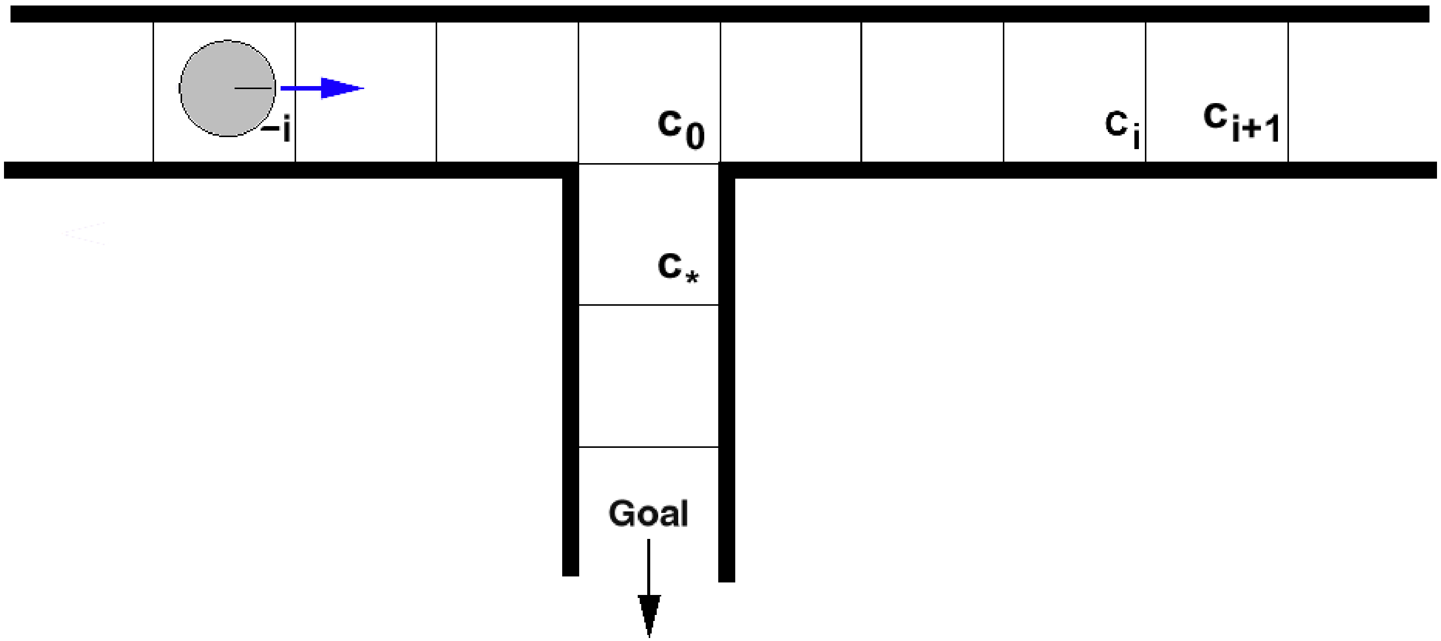
Problems of DWAs

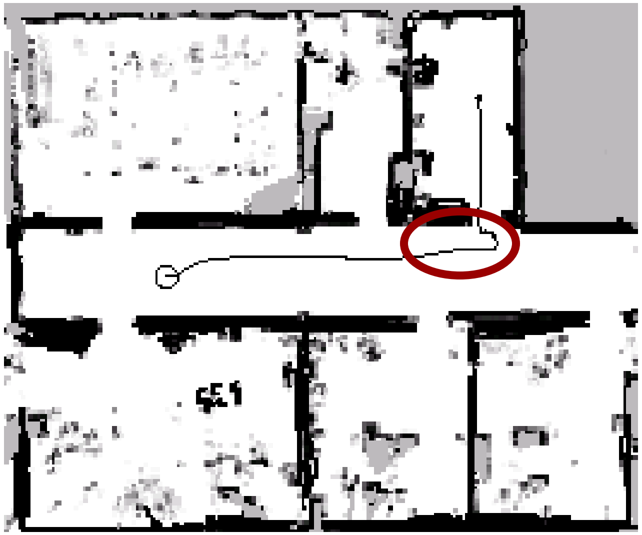


Same situation as in the beginning.

➔ DWAs have problems to reach the goal.

Problems of DWAs

- Typical problem in a real world situation:



- Robot does not slow down early enough to enter the doorway.

Motion Planning Formulation

- The **problem of motion planning** can be stated as follows. Given:
  - A **start** pose of the robot
  - A desired **goal** pose
  - A geometric description of the **robot**
  - A geometric representation of the **environment**

- Find a path that moves the robot gradually from **start** to **goal** while **never touching** any obstacle

Configuration Space

- Although the motion planning problem is defined in the regular world, it lives in another space: the **configuration space**

- A robot configuration q is a specification of the positions of all robot points relative to a fixed coordinate system

- Usually a configuration is expressed as a **vector of positions** and **orientations**

Configuration Space

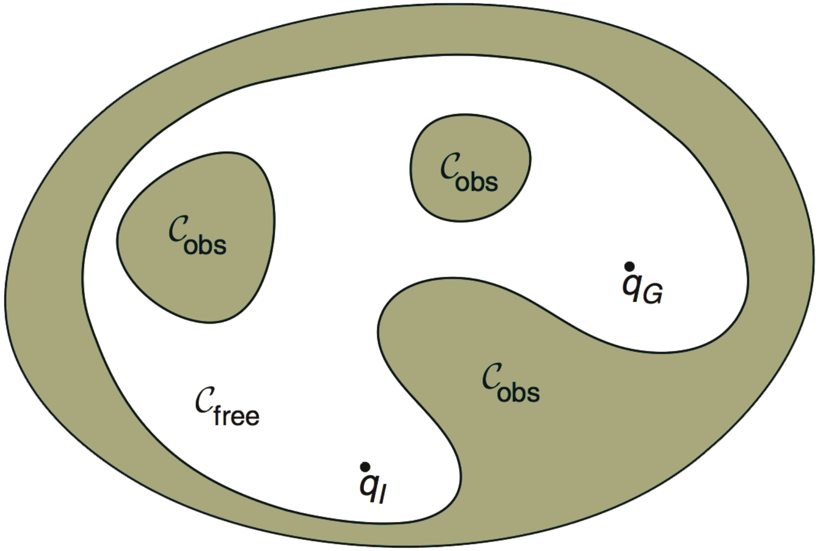
- **Free space** and **obstacle region**
- With $\mathcal{W} = \mathbb{R}^m$ being the work space, $\mathcal{O} \in \mathcal{W}$ the set of obstacles, $\mathcal{A}(q)$ the robot in configuration $q \in \mathcal{C}$

$$\begin{aligned} \mathcal{C}_{free} &= \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} = \emptyset\} \\ \mathcal{C}_{obs} &= \mathcal{C} / \mathcal{C}_{free} \end{aligned}$$

- We further define

  q_I : start configuration

  q_G : goal configuration



Configuration Space

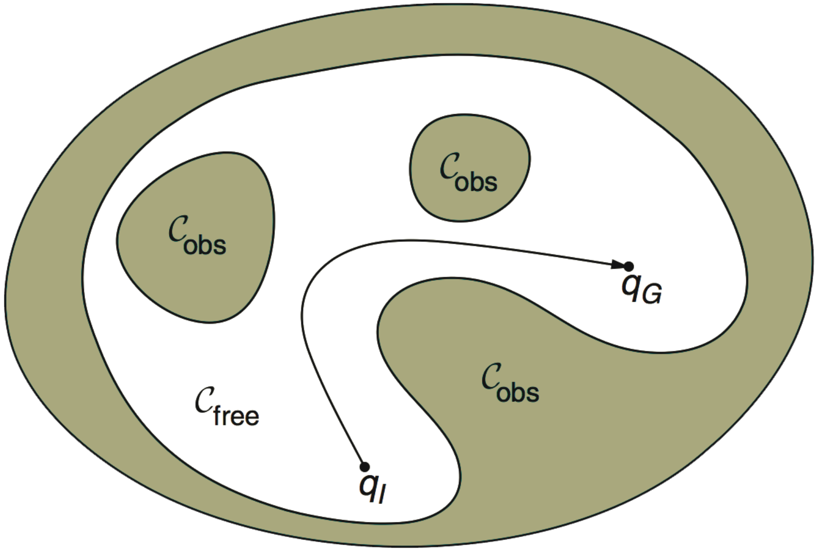
Then, motion planning amounts to

- Finding a continuous path

  $$\tau : [0,1] \to \mathcal{C}_{free}$$

  with $\tau(0) = q_I \, , \, \tau(1) = q_G$

- Given this setting, we can do planning with the robot being a **point in C-space!**


$\mathcal{C}_{\text{obs}}$
$\mathcal{C}_{\text{obs}}$
q_G
$\mathcal{C}_{\text{obs}}$
$\mathcal{C}_{\text{free}}$
q_I

C-Space Discretizations

- Continuous terrain needs to be **discretized** for path planning
- There are **two general approaches** to discretize C-spaces:
  - **Combinatorial planning**

    Characterizes C_{free} explicitly by capturing the connectivity of C_{free} into a graph and finds solutions using search
  - **Sampling-based planning**

    Uses collision-detection to probe and incrementally search the C-space for a solution

Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Uninformed search:** besides the problem definition, no further information about the domain (“blind search”)
- The only thing one can do is to expand nodes differently
- Example algorithms: breadth-first, uniform-cost, depth-first, bidirectional, etc.

Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

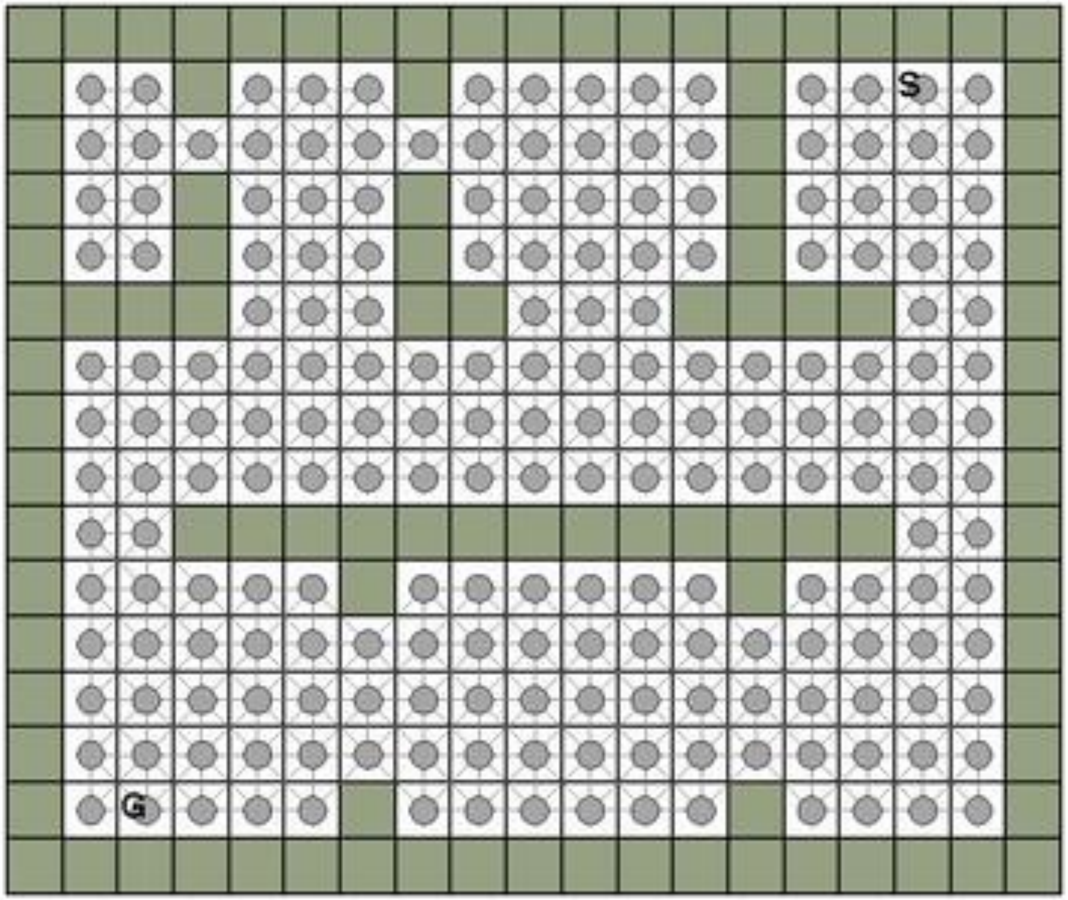
- **Informed search:** further information about the domain through **heuristics**
- Capability to say that a node is “more promising” than another node
- Example algorithms: greedy best-first search, **A***, many variants of A*, D*, etc.

Search

The performance of a search algorithm is measured in four different ways:

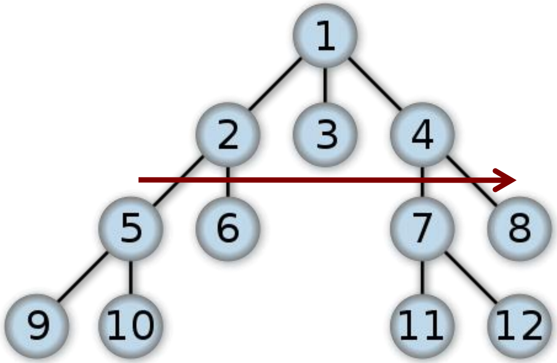
- **Completeness:** does the algorithm find a solution when there is one?
- **Optimality:** is the solution the best one of all possible solutions in terms of path cost?
- **Time complexity:** how long does it take to find a solution?
- **Space complexity:** how much memory is needed to perform the search?

Discretized Configuration Space

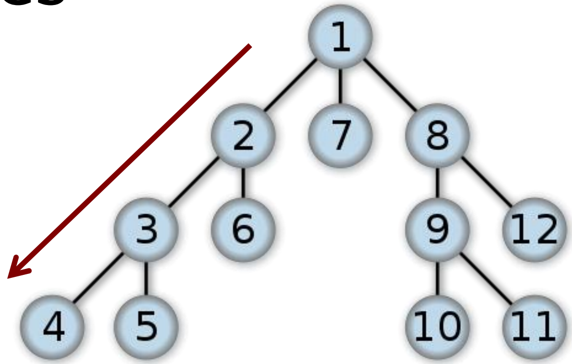


Uninformed Search

- **Breadth-first**
  - Complete
  - Optimal if action costs equal
  - Time and space: $O(b^d)$



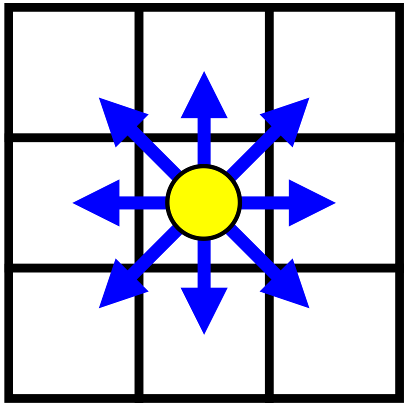
- **Depth-first**
  - Not complete in infinite spaces
  - Not optimal
  - Time: $O(b^m)$
  - Space: $O(bm)$ (can forget explored subtrees)



(b: branching factor, d: goal depth, m: max. tree depth)

Informed Search: A^*

- What about using A^* to plan the path of a robot?
- Finds the shortest path
- Requires a graph structure
- Limited number of edges
- In robotics: planning on a 2d occupancy grid map



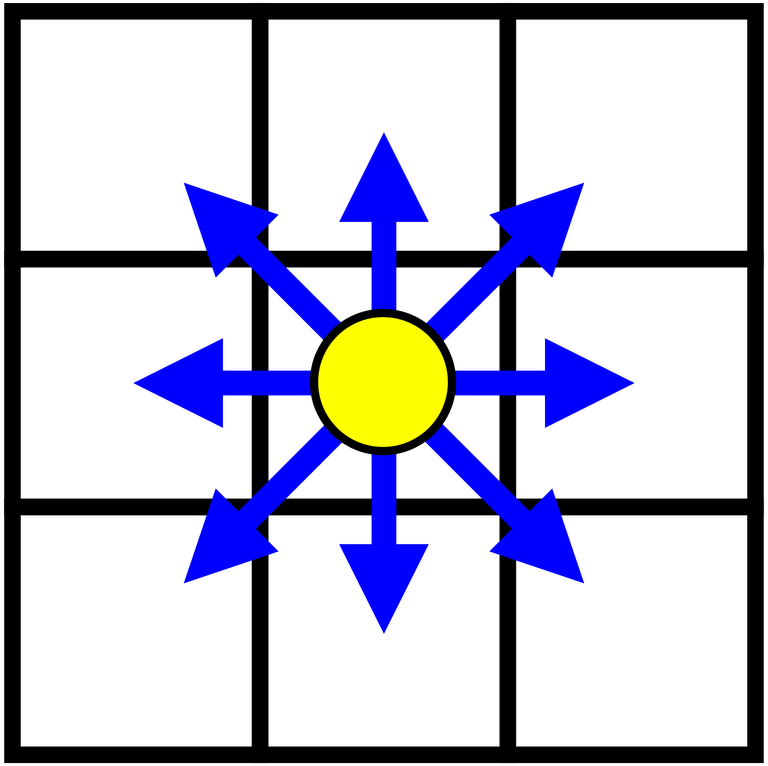
A*: Minimize the Estimated Path Costs

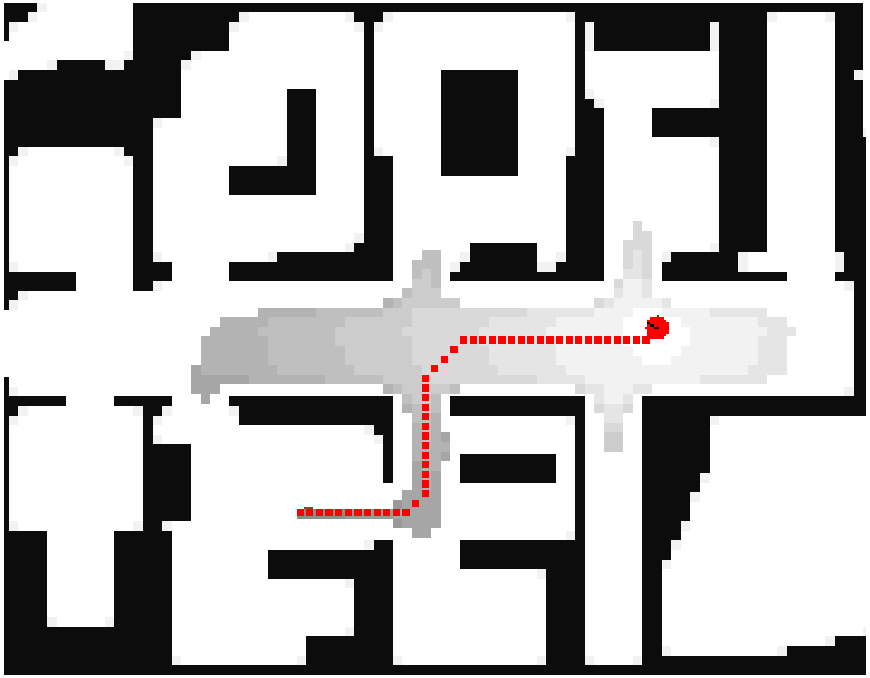
- *g(n)* = actual cost from the initial state to *n*.
- *h(n)* = estimated cost from *n* to the next goal.
- *f(n)* = *g(n)* + *h(n)*, the estimated cost of the cheapest solution through *n*.
- Let $h^*(n)$ be the actual cost of the optimal path from *n* to the next goal.
- *h* is admissible if the following holds for all *n* :

$$h(n) \leq h^*(n)$$

- We require that for A*, *h* is admissible (the straight-line distance is admissible in the Euclidean Space).

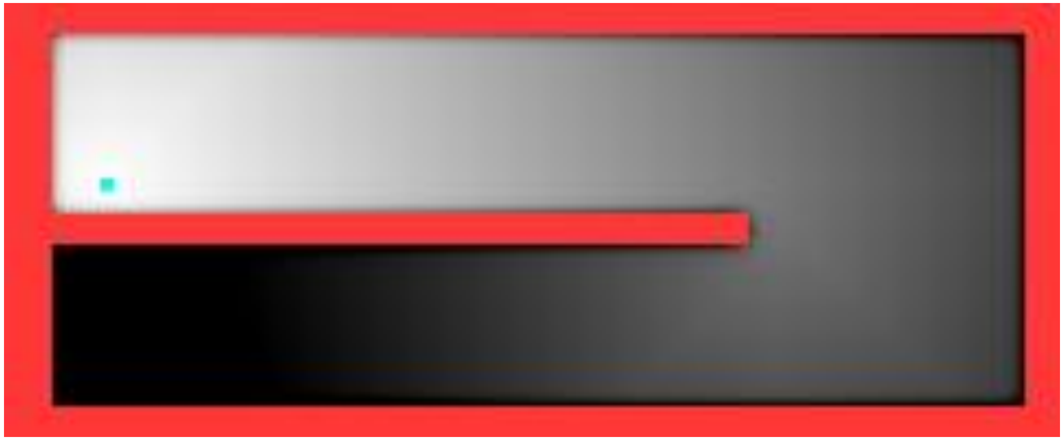
Example: PathPlanning for Robots in a Grid-World





Deterministic Value Iteration

- To compute the shortest path from every state to one goal state, use (deterministic) value iteration.
- Very similar to Dijkstra’s Algorithm.
- Such a cost distribution is the optimal heuristic for A*.



Typical Assumption in Robotics for A* Path Planning

1. The robot is assumed to be localized.
2. The robot computes its path based on an occupancy grid.
3. The correct motion commands are executed.

Are 1. and 3. always true?

Problems

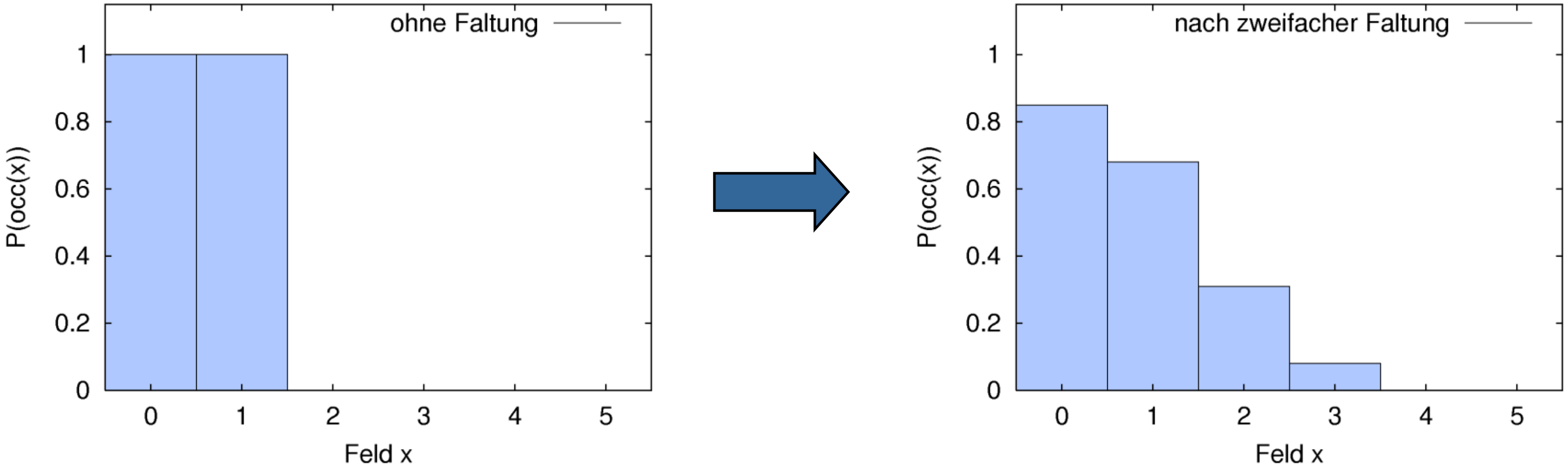
- What if the robot is (slightly) delocalized?
- Moving on the shortest path often guides the robot along a trajectory close to obstacles.
- Trajectory aligned to the grid structure.

Convolution of the Grid Map

- Convolution blurs the map.
- Obstacles are assumed to be bigger than in reality.
- Perform an A^* search in such a convolved map (using occupancy as traversal cost).
- Robot increases distance to obstacles and moves on a short path!

Example: Map Convolution

- one-dimensional environment, cells $c_0, \ldots, c_5$



- Cells before and after 2 convolution runs.

Convolution

- Consider an occupancy map. Than the convolution is defined as:

$$P(occ_{x_i,y}) = \frac{1}{4} \cdot P(occ_{x_{i-1},y}) + \frac{1}{2} \cdot P(occ_{x_i,y}) + \frac{1}{4} \cdot P(occ_{x_{i+1},y})$$

$$P(occ_{x_0,y}) = \frac{2}{3} \cdot P(occ_{x_0,y}) + \frac{1}{3} \cdot P(occ_{x_1,y})$$

$$P(occ_{x_{n-1},y}) = \frac{1}{3} \cdot P(occ_{x_{n-2},y}) + \frac{2}{3} \cdot P(occ_{x_{n-1},y})$$

- This is done for each row and each column of the map.
- “Gaussian blur”

A* in Convolved Maps

- The costs are a product of path length and occupancy probability of the cells.
- Cells with higher probability (e.g., caused by convolution) are avoided by the robot.
- Thus, it keeps distance to obstacles.
- This technique is fast and quite reliable.

5D-Planning – an Alternative to the Two-layered Architecture

- Plans in the full <x,y,θ,v,ω>-configuration space using A^*.
  - ➔ Considers the robot's kinematic constraints.
- Generates a sequence of steering commands to reach the goal location.
- Maximizes trade-off between driving time and distance to obstacles.

The Search Space (1)

- What is a state in this space?
  $<x,y,\theta,v,\omega>$ = current position and speed of the robot

- How does a state transition look like?
  $<x1,y1,\theta1,v1,\omega1> \rightarrow <x2,y2,\theta2,v2,\omega2>$
  - with motion command $(v2,\omega2)$ and $|v1-v2| < av$, $|\omega1-\omega2| < a\omega$.
  - The new pose of the Robot $<x2,y2,\theta2>$ is a result of the motion equations.

The Search Space (2)

Idea: search in the discretized $\langle x,y,\theta,v,\omega \rangle$-space.

Problem: the search space is too huge to be explored within the time constraints (5+ Hz for online motion planning).

Solution: restrict the full search space.

The Main Steps of the Algorithm

1. Update (static) grid map based on sensory input.

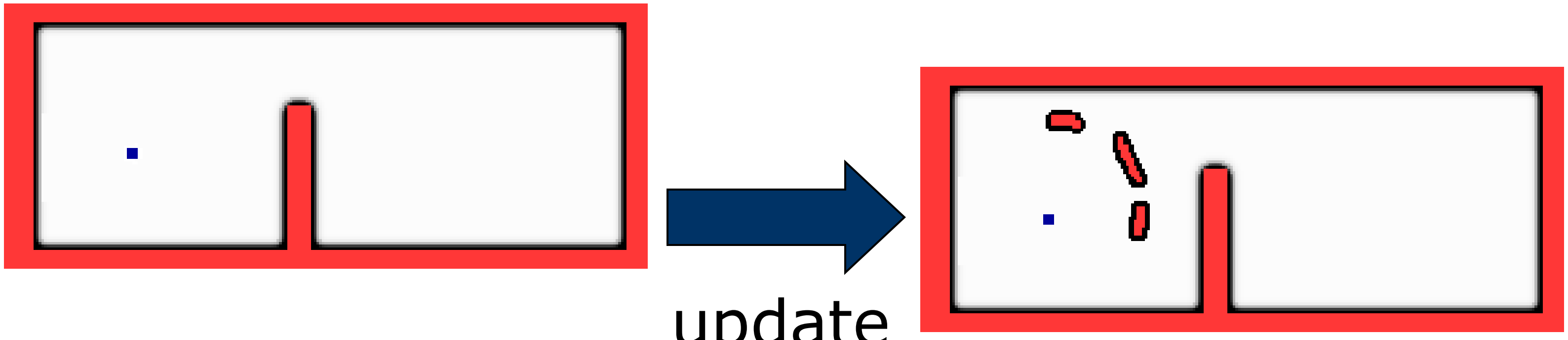
2. Use A^* to find a trajectory in the <x,y>-space using the updated grid map.

3. Determine a restricted 5d-configuration space based on step 2.

4. Find a trajectory by planning in the restricted $\langle x,y,\theta,v,\omega \rangle$-space.

Updating the Grid Map

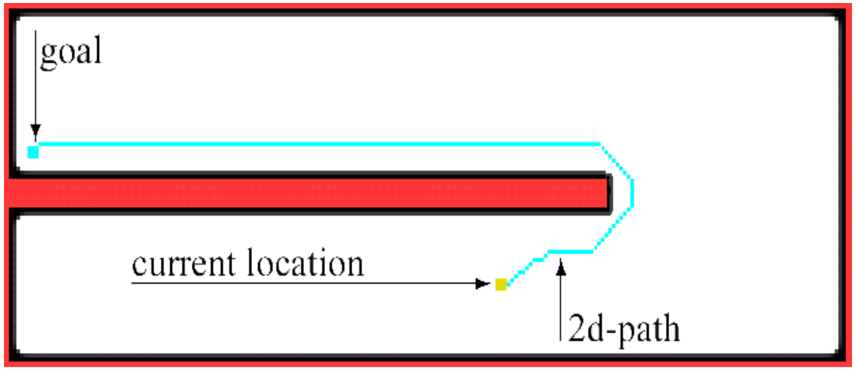
- The environment is represented as a 2d-occupency grid map.
- Convolution of the map increases security distance.
- Detected obstacles are added.
- Cells discovered free are cleared.

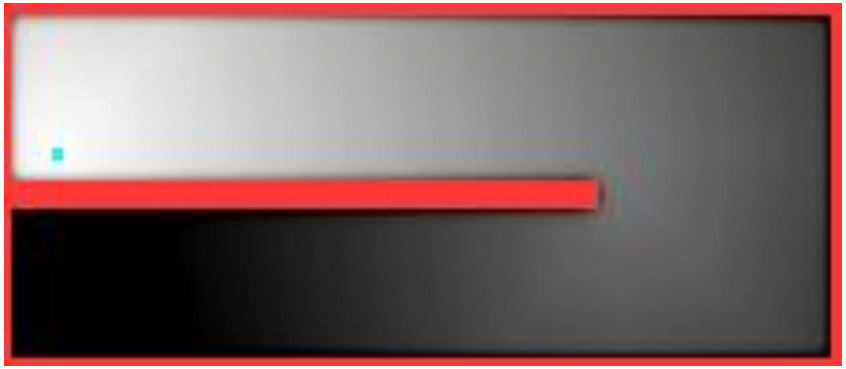


Find a Path in the 2d-Space

- Use A* to search for the optimal path in the 2d-grid map.

- Use heuristic based on a deterministic value iteration within the static map.





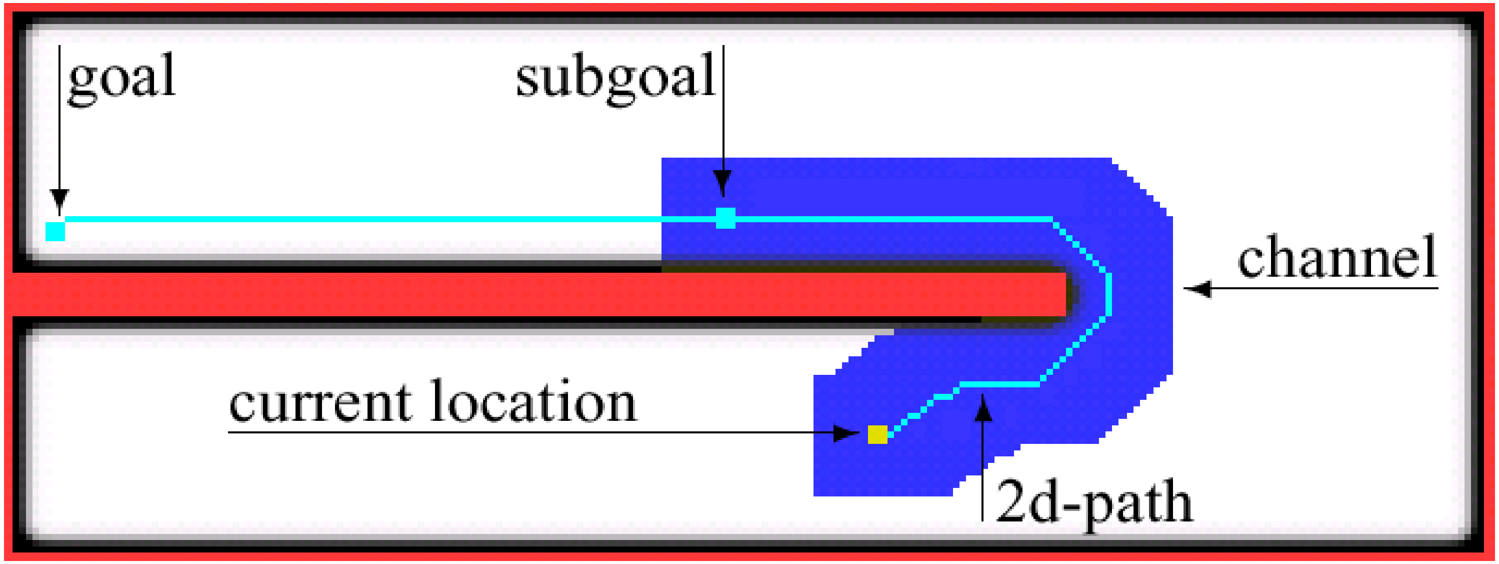
Restricting the Search Space

Assumption: the projection of the 5d-path onto the <x,y>-space lies close to the optimal 2d-path.

Therefore: construct a restricted search space (channel) based on the 2d-path.

Space Restriction

- Resulting search space = $\langle x, y, \theta, v, \omega \rangle$ with $(x,y) \in$ channel.
- Choose a sub-goal lying on the 2d-path within the channel.

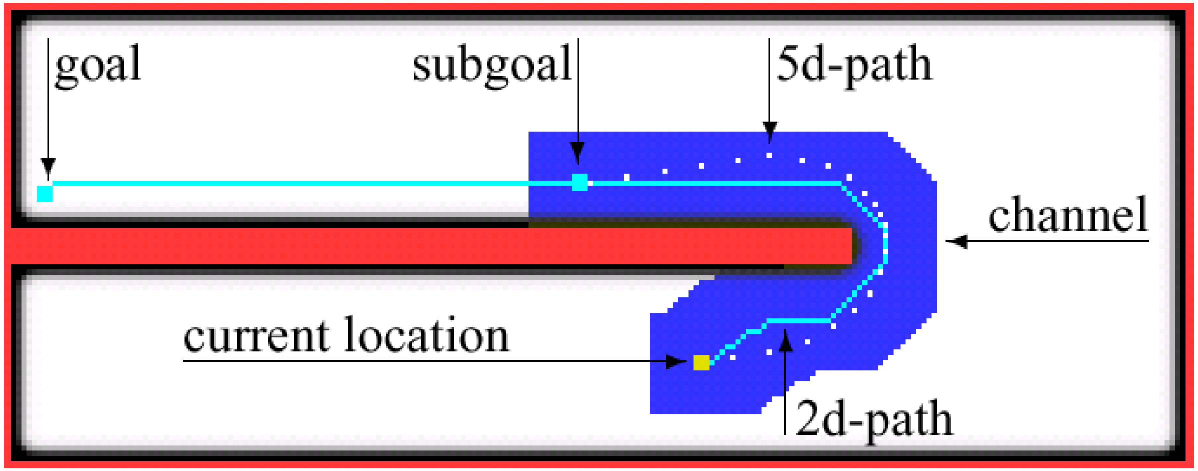


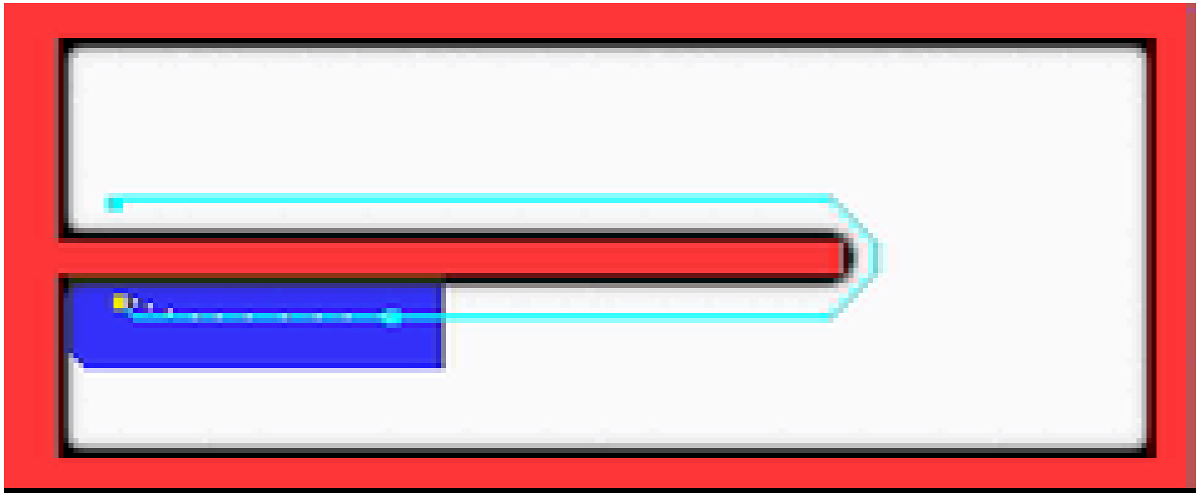
Find a Path in the 5d-Space

- Use A* in the restricted 5d-space to find a sequence of steering commands to reach the sub-goal.

- To estimate cell costs: perform a deterministic 2d-value iteration within the channel.

Examples





Timeouts

- Steering a robot online requires to set new steering commands frequently.
E.g., every 0.2 secs.

➔ Abort search after 0.2 secs.

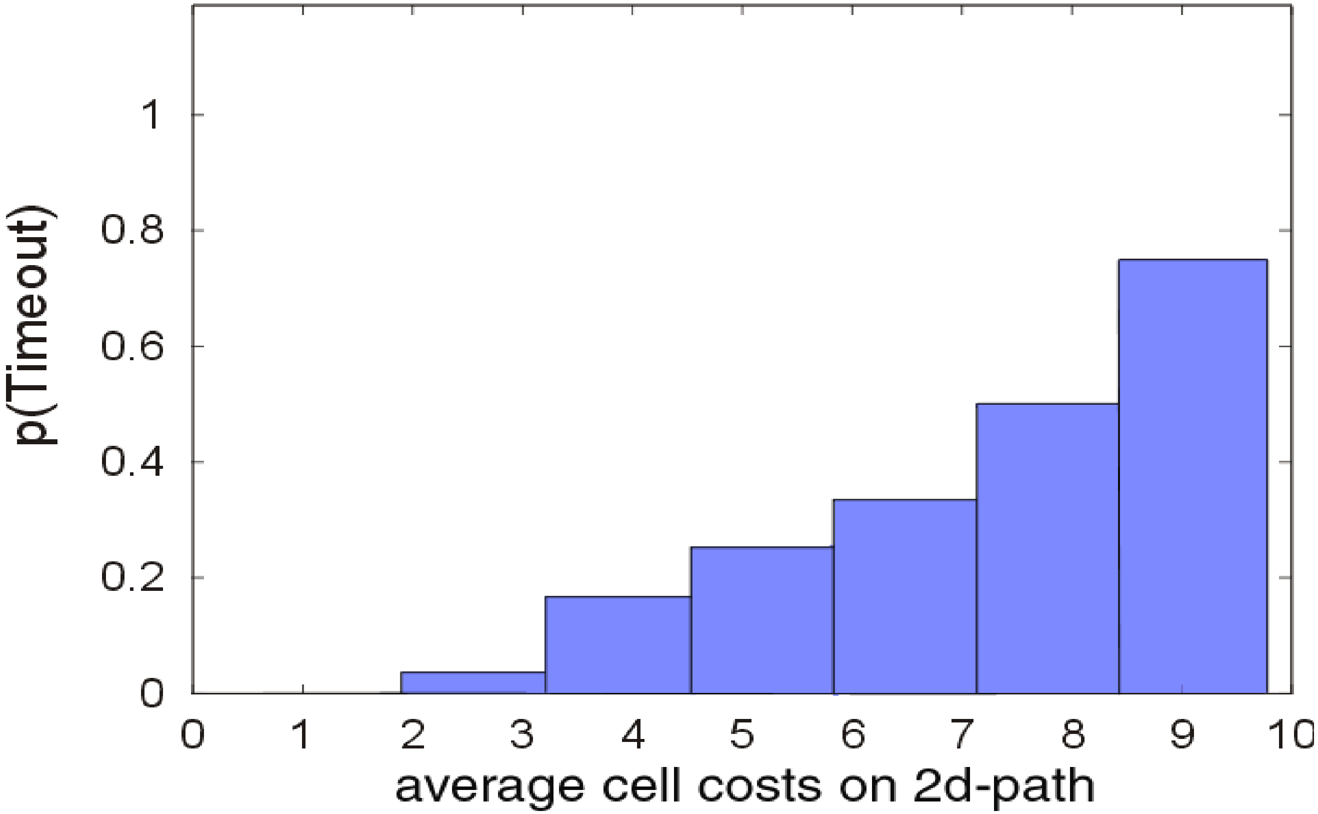
How to find an admissible steering command?

Alternative Steering Command

- Previous trajectory still admissible?
  → **OK**

- If not, drive on the 2d-path or use DWA to find new command.

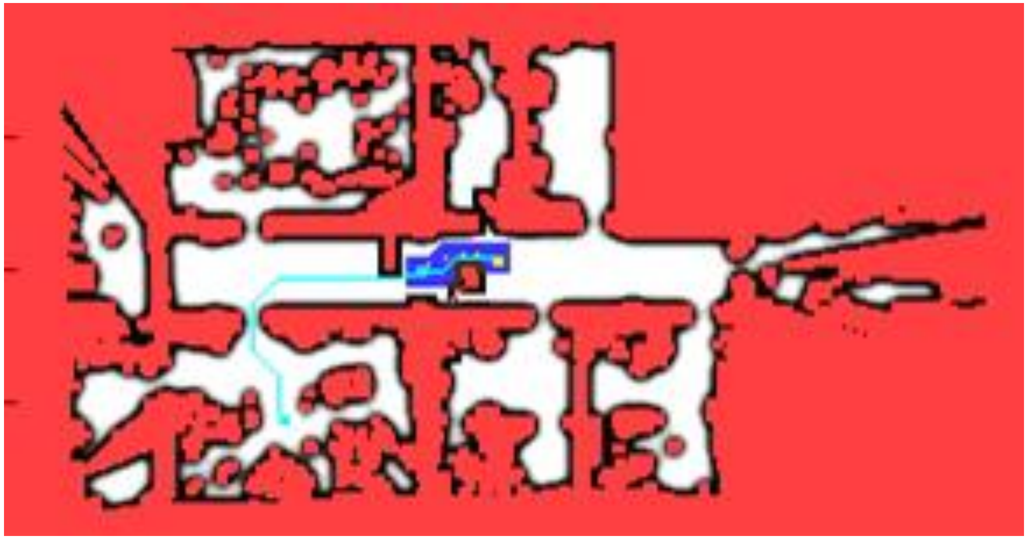
Timeout Avoidance



➔ Reduce the size of the channel if the 2d-path has high cost.

Example

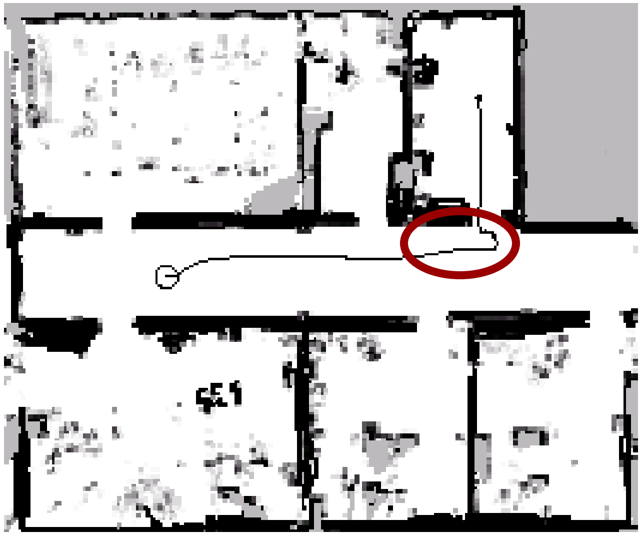
Robot Albert



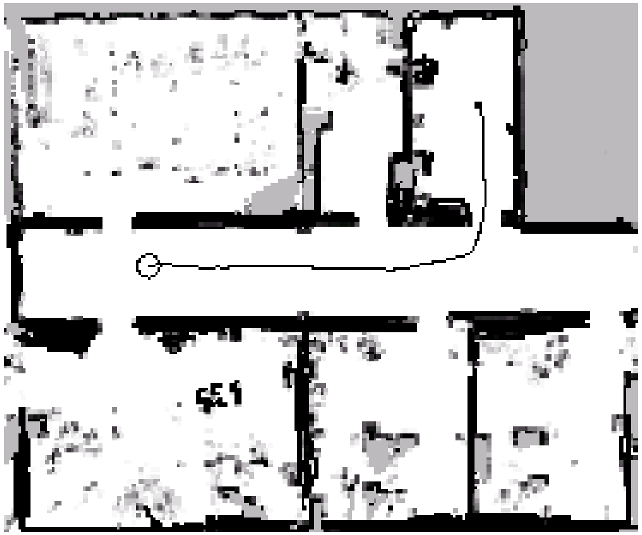
Planning state

Comparison to the DWA (1)

- DWAs often have problems entering narrow passages.

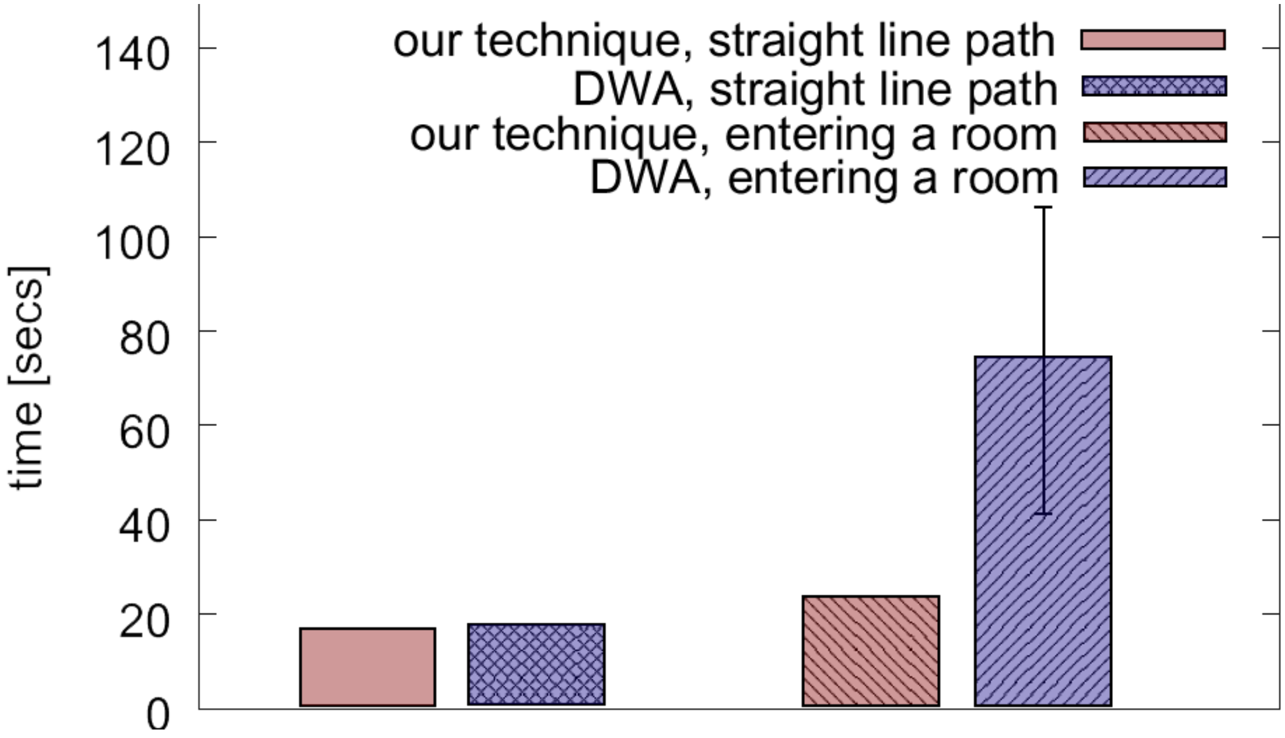


DWA planned path.



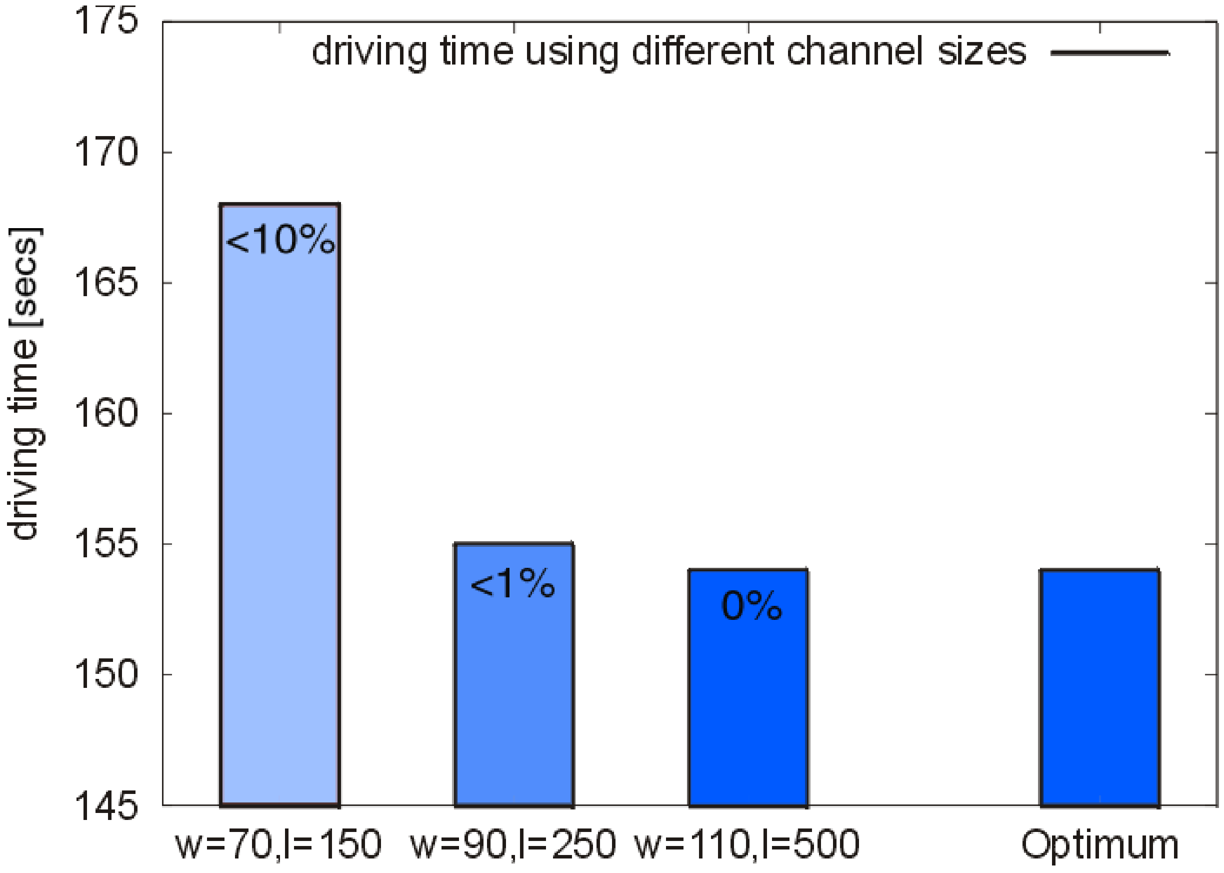
5D approach.

Comparison to the DWA (2)



The 5D approach results in significantly faster motion when driving through narrow passages!

Comparison to the Optimum

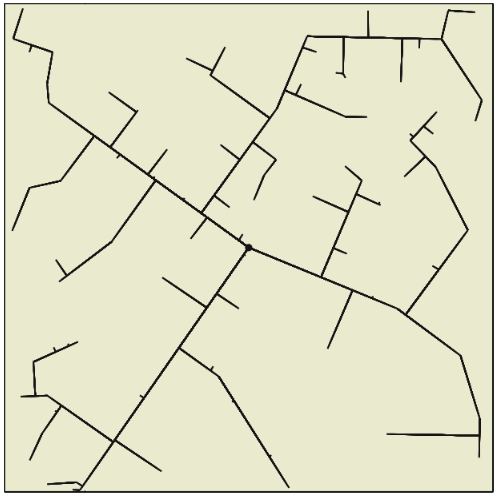


Channel: with length=5m, width=1.1m

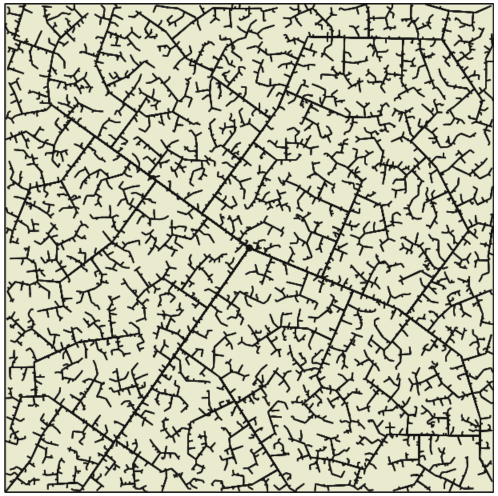
Resulting actions are close to the optimal solution.

Rapidly Exploring Random Trees

- **Idea:** aggressively probe and explore the C-space by **expanding incrementally** from an initial configuration q_0
- The explored territory is marked by a **tree rooted at q_0**



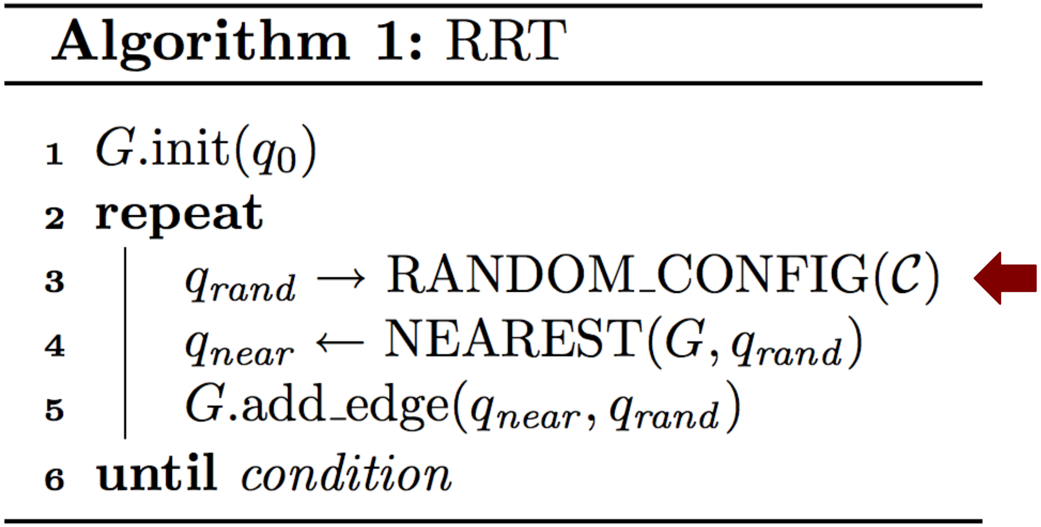
45 iterations



2,345 iterations

RRTs

The algorithm: Given C and q_0

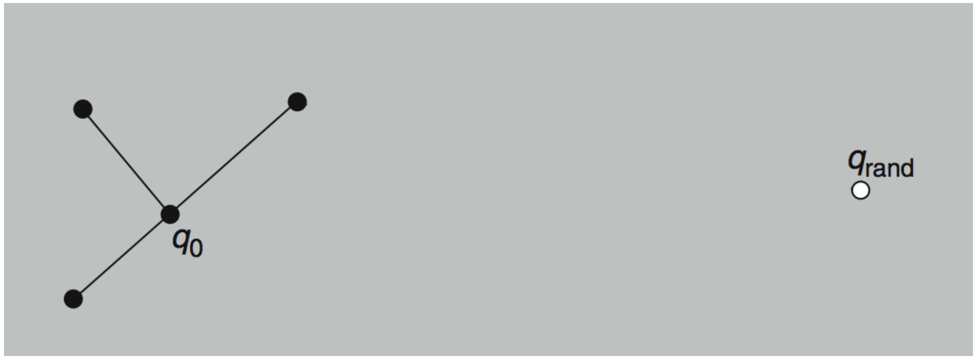


Algorithm 1: RRT

```
$G$.init($q_0$)
repeat
    $q_{rand} \rightarrow$ RANDOM_CONFIG($\mathcal{C}$)
    $q_{near} \leftarrow$ NEAREST($G, q_{rand}$)
    $G$.add_edge($q_{near}, q_{rand}$)
until condition
```

Sample from a **bounded region** centered around q_0

E.g. an axis-aligned relative random translation or random rotation



RRTs

- The algorithm

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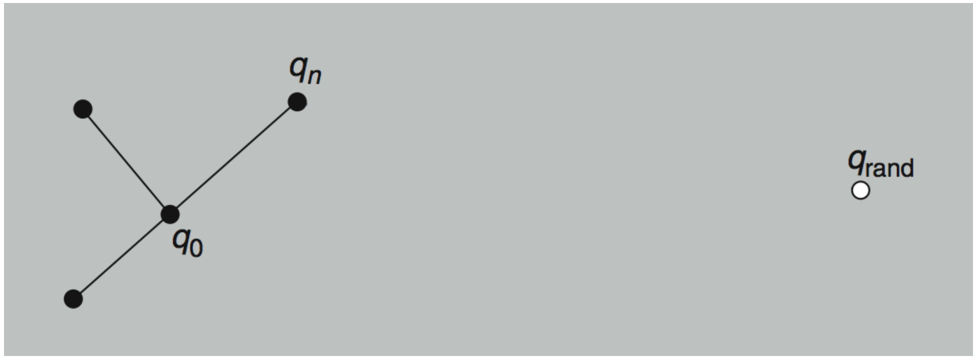
until condition
```

Finds closest vertex in G using a **distance function**

$$\rho \;:\; \mathcal{C} \times \mathcal{C} \longrightarrow [0, \infty)$$

formally a ***metric*** defined on C



RRTs

- The algorithm

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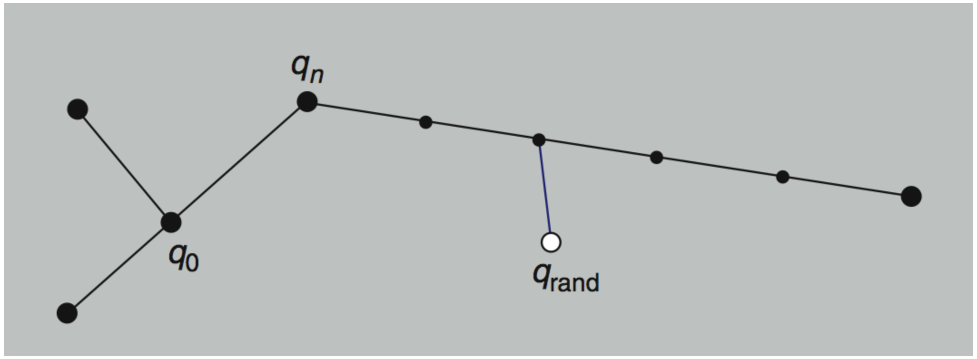
until condition
```

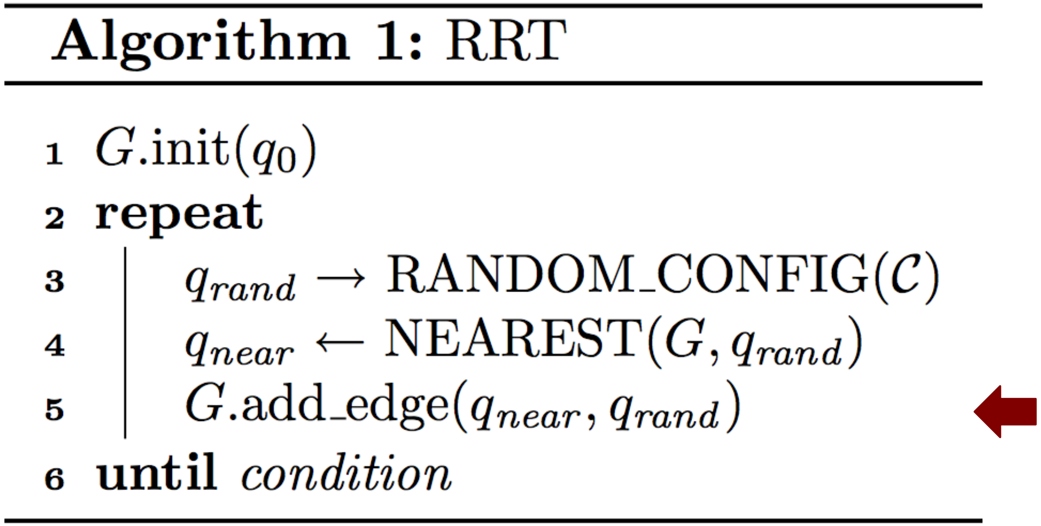
Several stategies to find q_{near} given the closest vertex on G:

- Take closest vertex
- Check intermediate points at regular intervals and split edge at q_{near}



RRTs

- The algorithm



Algorithm 1: RRT

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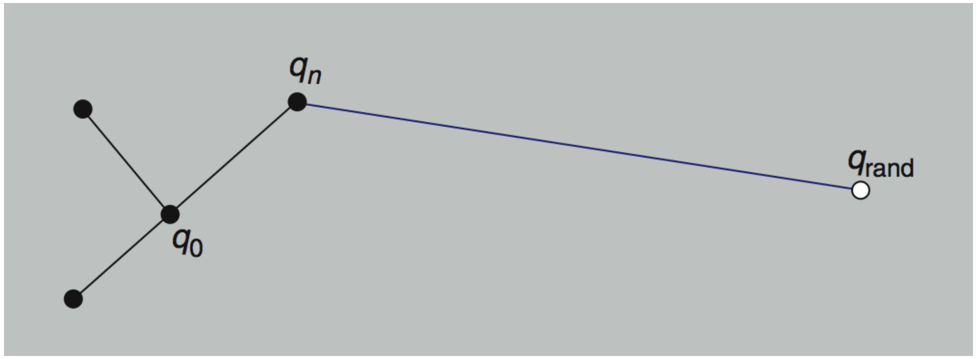
    $G$.add_edge($q_{near}, q_{rand}$)
until condition
```

Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}

- No collision: add edge



RRTs

- The algorithm

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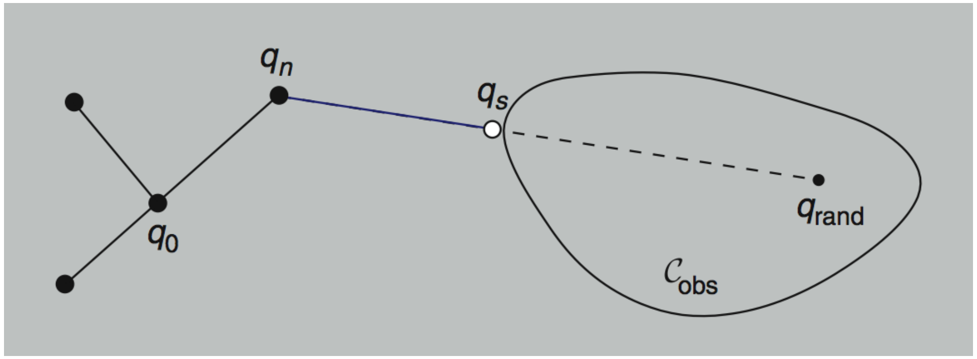


- No collision: add edge
- Collision: new vertex is q_s, as close as possible to C_{obs}

RRTs

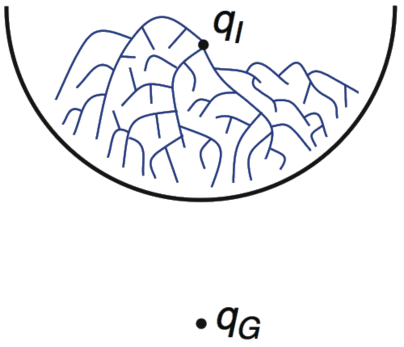
- How to perform path planning with RRTs?

  1. Start RRT at q_I
  2. At every, say, 100th iteration, force $q_{rand} = q_G$
  3. If q_G is reached, problem is solved

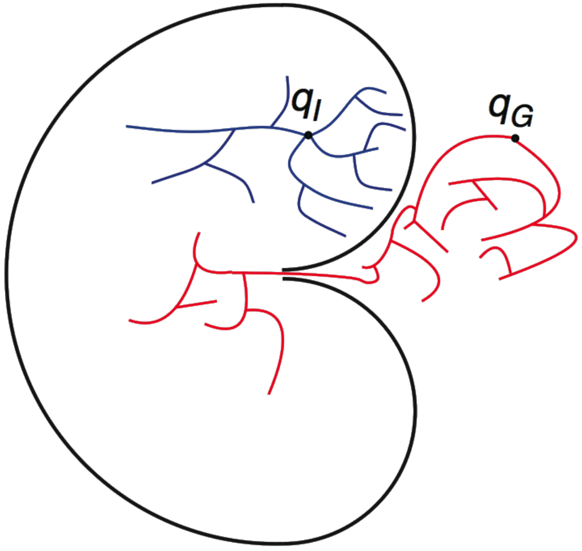
- Why not picking q_G every time?
- This will fail and waste much effort in running into C_{Obs} instead of exploring the space

RRTs

- However, some problems require more effective methods: **bidirectional search**
- Grow **two** RRTs, one from q_I, one from q_G
- In every other step, try to extend each tree towards the newest vertex of the other tree



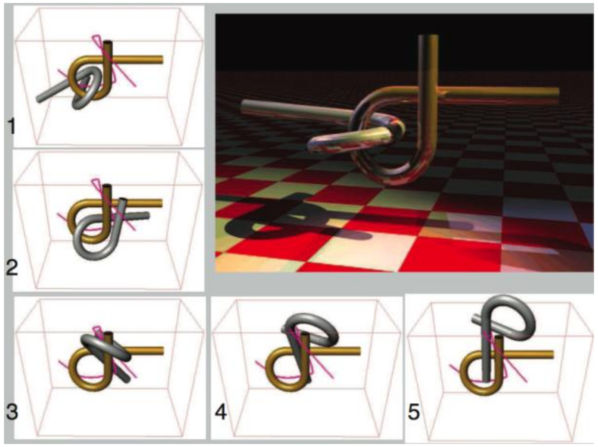
Filling a well



A bug trap

RRTs

- RRTs are popular, many extensions exist: real-time RRTs, anytime RRTs, for dynamic environments etc.
- **Pros:**
  - Balance between greedy search and exploration
  - Easy to implement
- **Cons:**
  - Metric sensivity
  - Unknown rate of convergence



Alpha 1.0 puzzle. Solved with bidirectional RRT

Road Map Planning

- A **road map** is a **graph in $\boldsymbol{C_{free}}$** in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

- Several **planning techniques**
  - Visibility graphs
  - Voronoi diagrams
  - Exact cell decomposition
  - Approximate cell decomposition
  - Randomized road maps

Road Map Planning

- A **road map** is a **graph in** $C_{\boldsymbol{free}}$ in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

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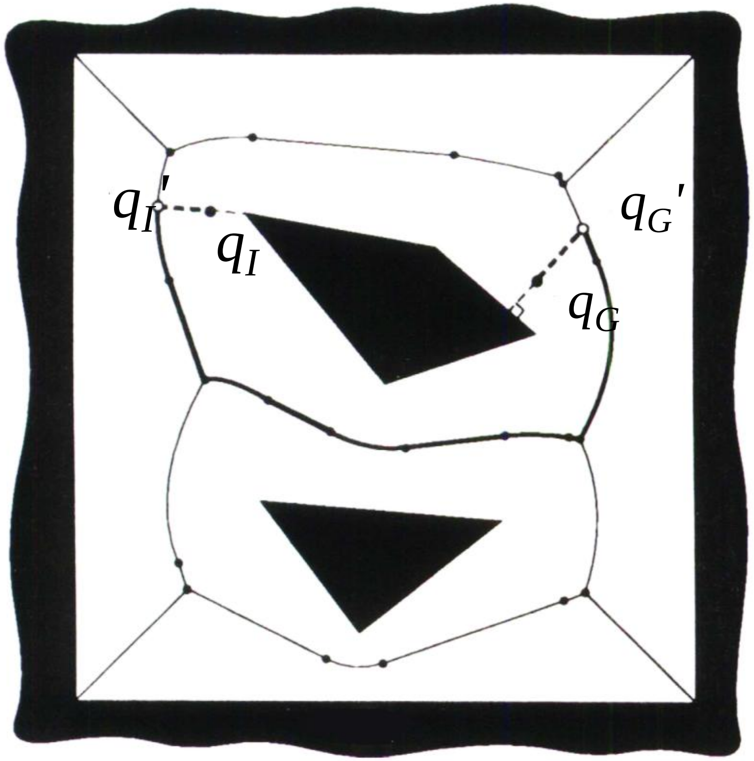
  - Visibility graphs
  - **Voronoi diagrams**
  - Exact cell decomposition
  - Approximate cell decomposition
  - **Randomized road maps**

Generalized Voronoi Diagram

- **Defined** to be the set of points q whose cardinality of the set of boundary points of C_{obs} with the same distance to q is greater than 1

- Let us decipher this definition...

- **Informally:** the place with the same **maximal clearance** from all nearest obstacles



Generalized Voronoi Diagram

- **Formally:**

  Let $\beta = \partial \mathcal{C}_{free}$ be the boundary of C_{free}, and $d(p,q)$ the Euclidian distance between p and q. Then, for all q in C_{free}, let

  $$clearance(q) = \min_{p \in \beta} d(p, q)$$

  be the *clearance* of q, and

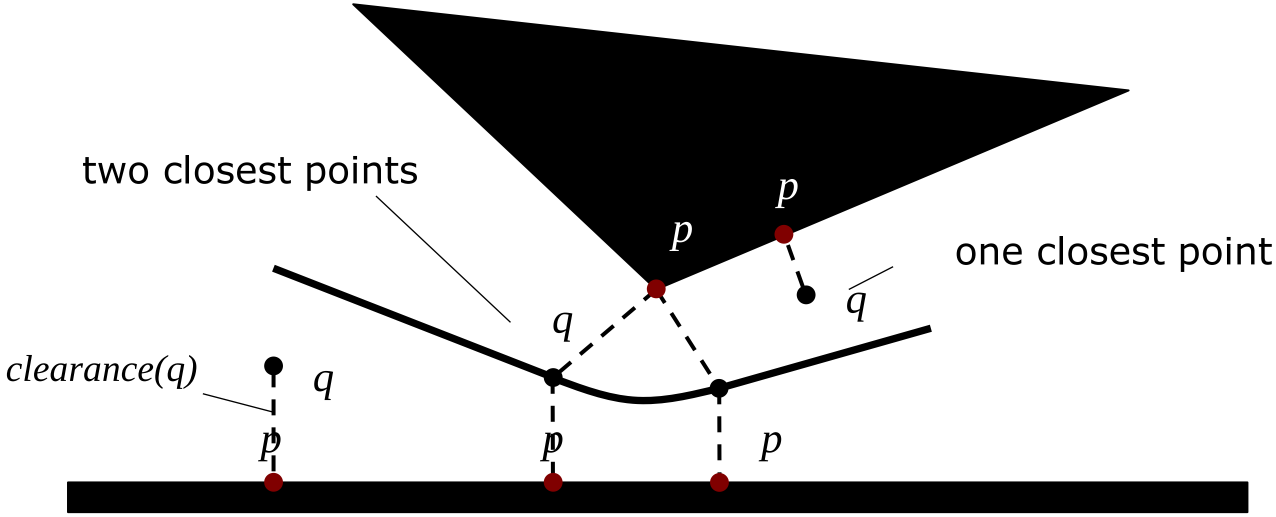
  $$near(q) = \{p \in \beta \mid d(p, q) = clearance(q)\}$$

  the set of "base" points on β with the same clearance to q. The **Voronoi diagram** is then the set of q's with more than one base point p

  $$V(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} \mid |near(q)| > 1\}$$

Generalized Voronoi Diagram

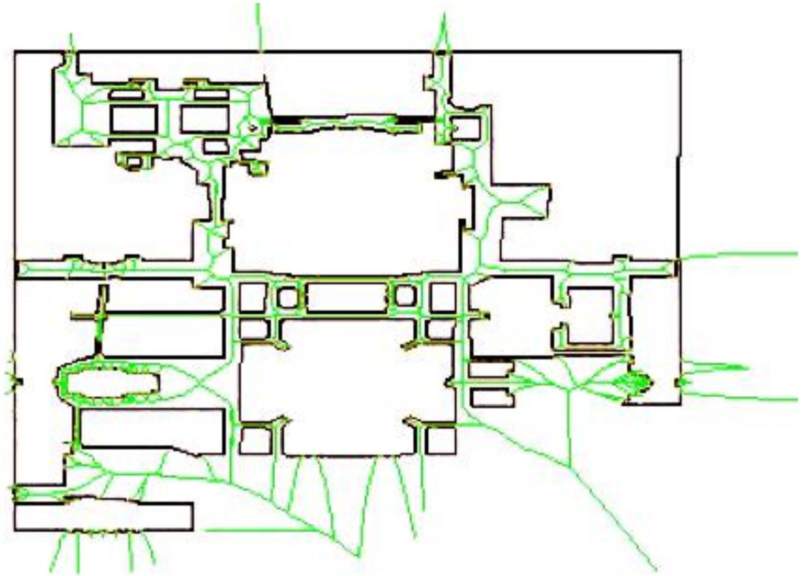
- **Geometrically:**



- For a polygonal C_{obs}, the Voronoi diagram consists of (n) lines and parabolic segments
- Naive algorithm: $O(n^4)$, best: $O(n \log n)$

Voronoi Diagram

- Voronoi diagrams have been well studied for (reactive) **mobile robot** path planning
- Fast methods exist to compute and update the diagram in real-time for low-dim. C's
  - **Pros:** maximize clearance is a good idea for an uncertain robot
  - **Cons:** unnatural attraction to open space, suboptimal paths



- Needs extensions

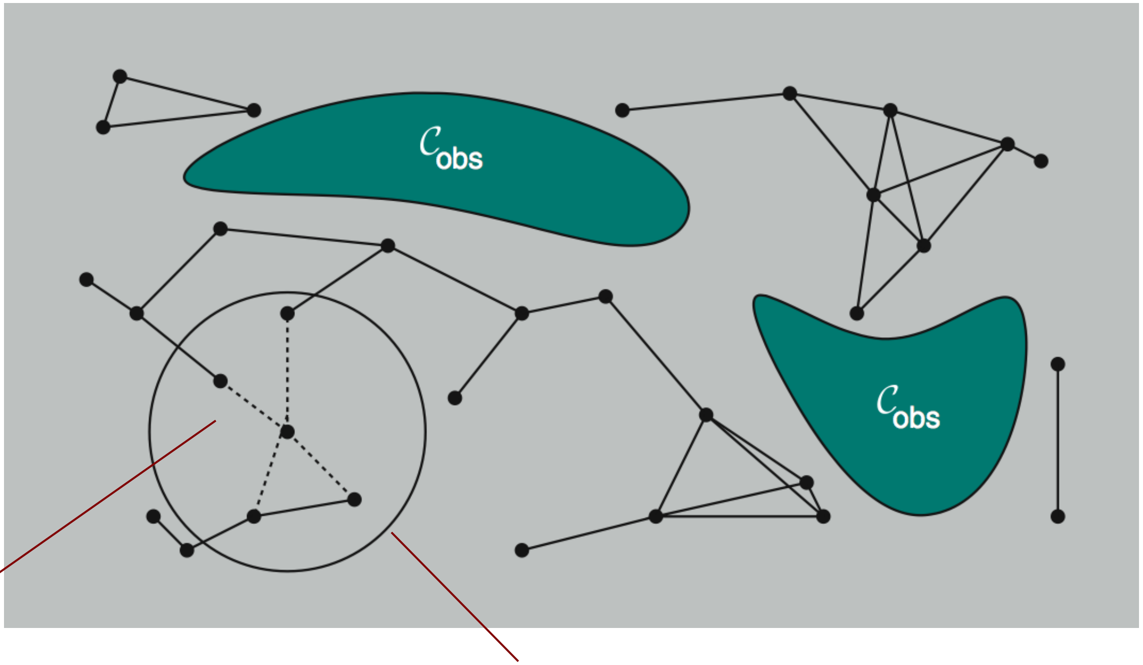
Randomized Road Maps

Also called *Probabilistic Road Maps*

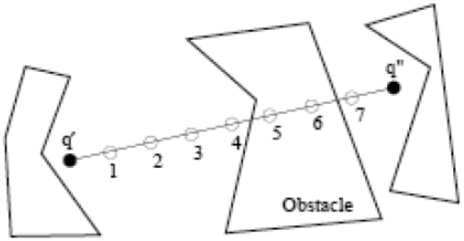
- **Idea:** Take random samples from C, declare them as vertices if in C_{free}, try to connect nearby vertices with local planner
- The **local planner** checks if line-of-sight is collision-free (powerful or simple methods)
- Options for *nearby:* **k-nearest neighbors** or all neighbors within **specified radius**
- Configurations and connections are added to graph until roadmap is **dense enough**

Randomized Road Maps

- Example



specified
radius

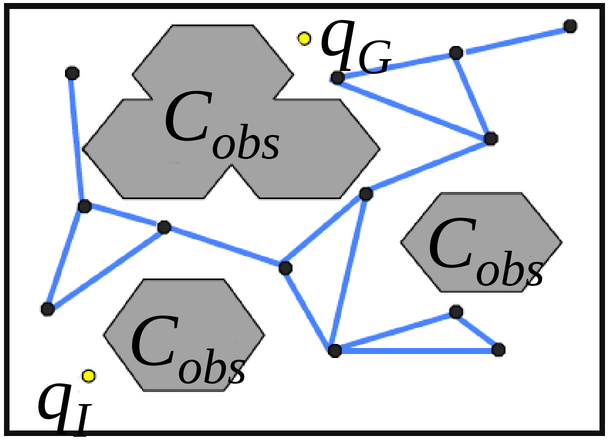


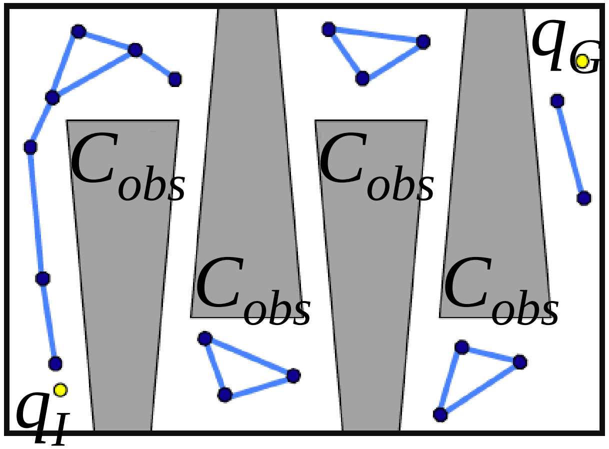
Example local
planner

What does “nearby” mean
on a manifold? Defining a
good metric on C is crucial

Randomized Road Maps

- **Pros:**
  - *Probabilistically complete*
  - Do not construct C-space
  - Apply easily to high dimensional C-spaces
  - Randomized road maps have solved previously unsolved problems
- **Cons:**
  - Do not work well for some problems, narrow passages
  - Not optimal, not complete





Randomized Road Maps

- How to **uniformly sample** C ? This is not at all trivial given its topology
- For example over spaces of rotations: Sampling Euler angles gives more samples near poles, not uniform over $SO(3)$. Use quaternions!
- However, Randomized Road Maps are **powerful, popular** and **many extensions** exist: advanced sampling strategies (e.g. near obstacles), PRMs for deformable objects, closed-chain systems, etc.

From Road Maps to Paths

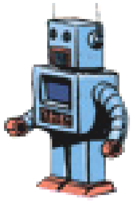
- All methods discussed so far **construct a road map** (without considering the query pair q_I and q_G)

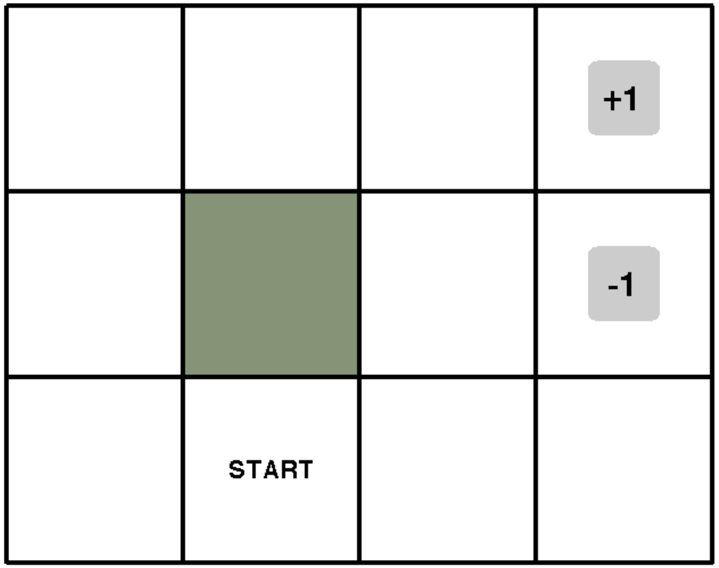
- Once the investment is made, the **same road map** can be reused for **all** queries (provided world and robot do not change)

1. **Find** the cell/vertex that contain/is close to q_I and q_G (not needed for visibility graphs)
2. **Connect** q_I and q_G to the road map
3. **Search** the road map for a path from q_I to q_G

Markov Decision Process

- Consider an agent acting in this environment

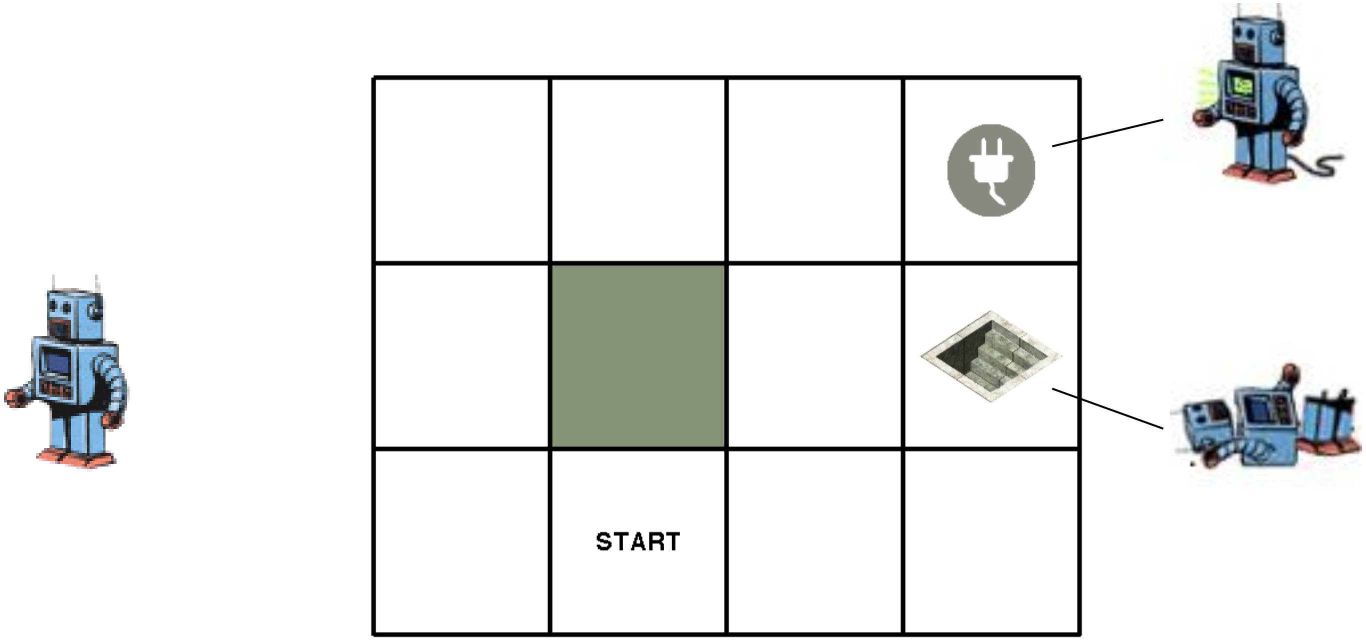




- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

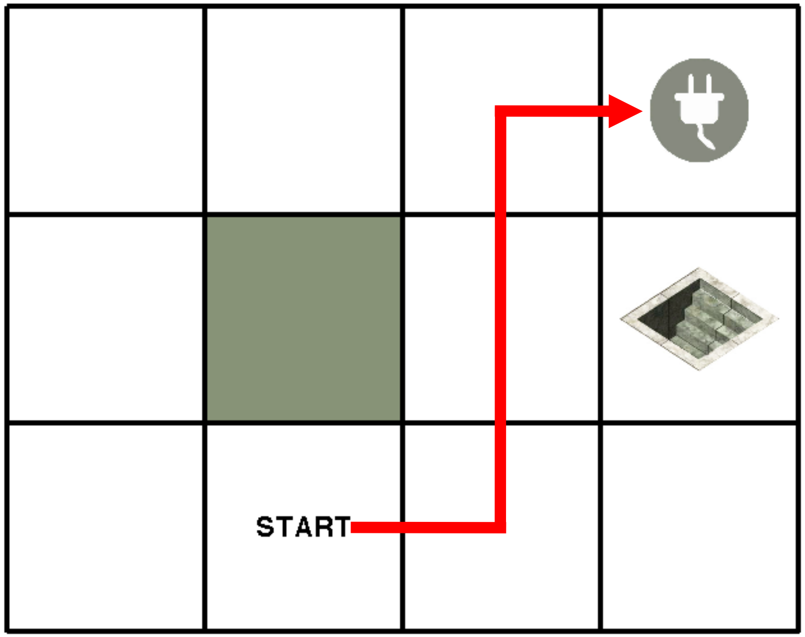
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

- Easy! Use a search algorithm such as A*



- Best solution (shortest path) is the action sequence *[Right, Up, Up, Right]*

What is the problem?

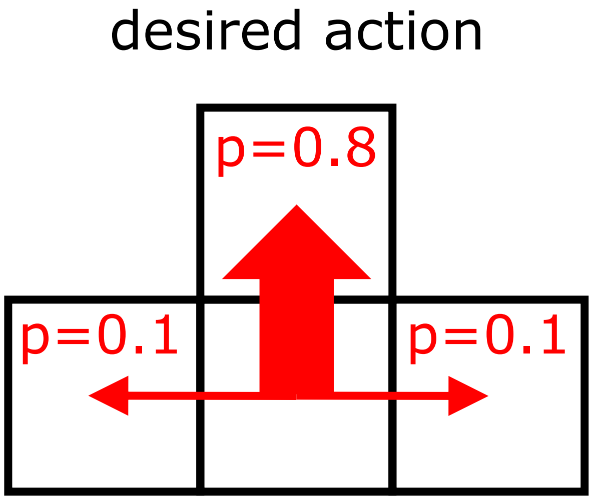
- Consider a non-perfect system in which actions are performed with a **probability less than 1**
- What are the best actions for an agent under this constraint?

- Example: a mobile robot does not *exactly* perform a desired motion
- Example: human navigation

➡ Uncertainty about performing actions!

MDP Example

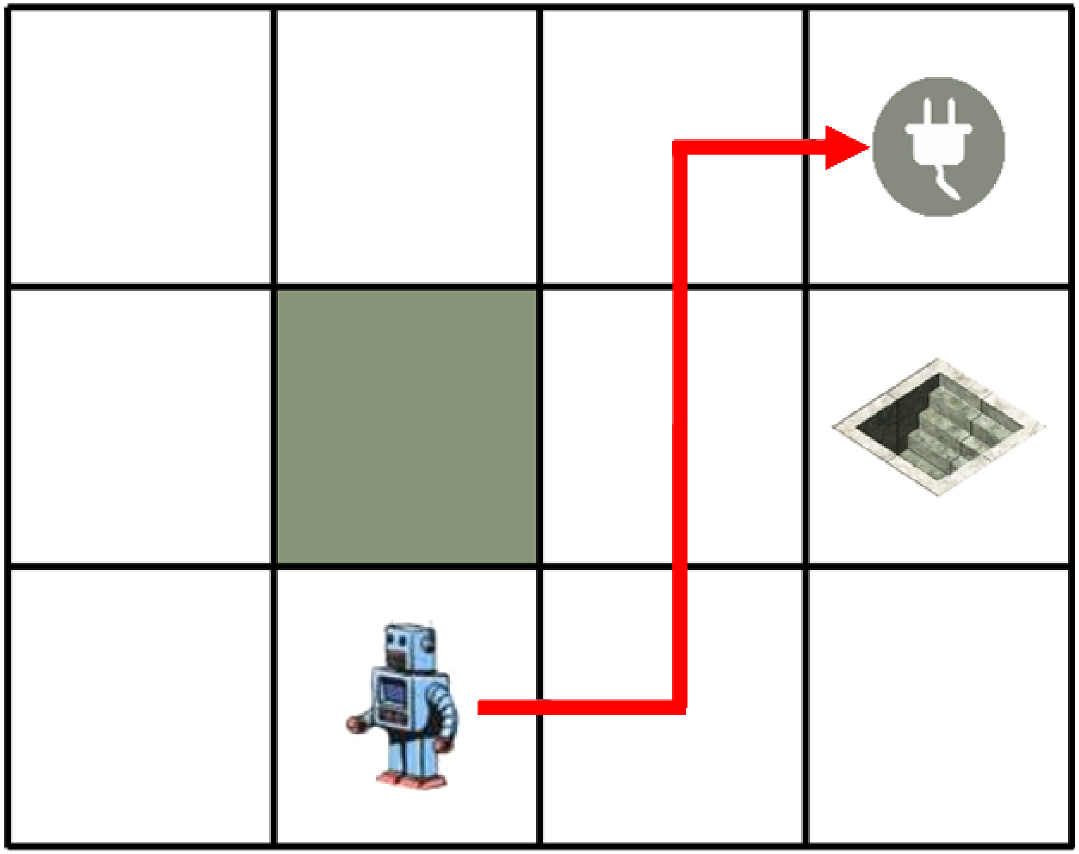
- Consider the **non-deterministic transition model** (N / E / S / W):



- Intended action is executed with p=0.8
- With p=0.1, the agent moves left or right
- Bumping into a wall “reflects” the robot

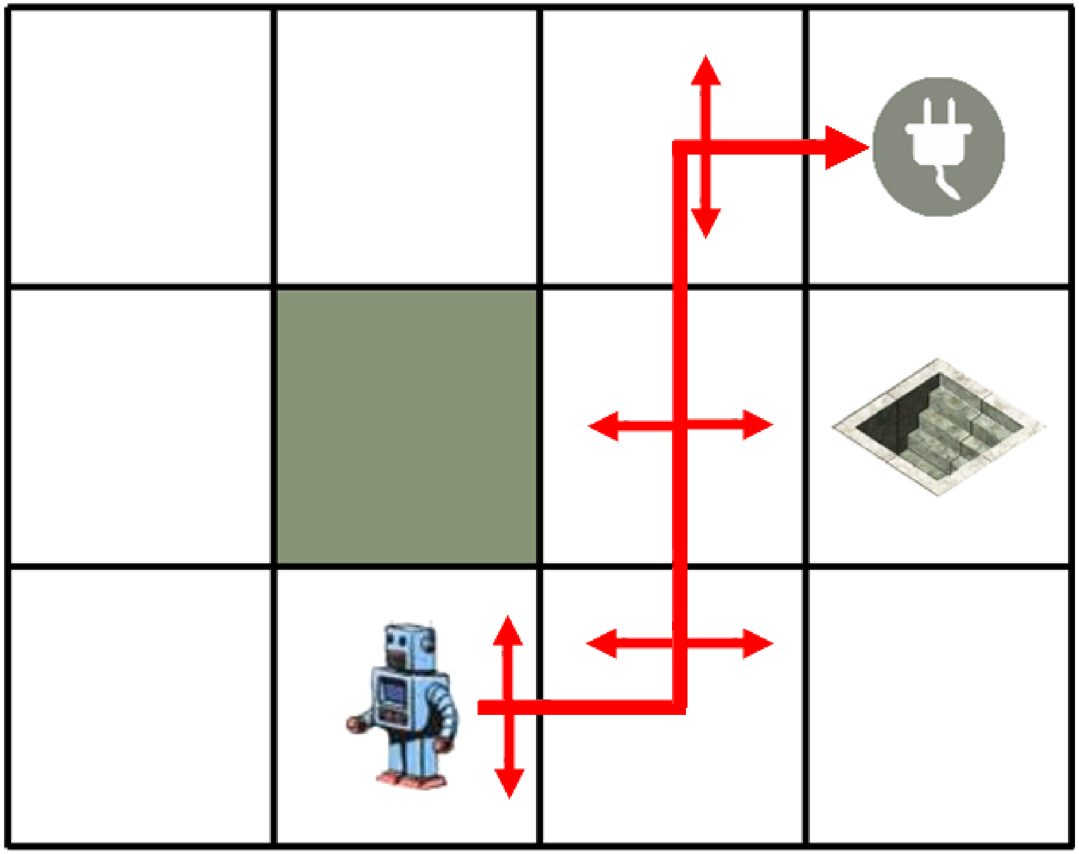
MDP Example

- Executing the **A* plan** in this environment



MDP Example

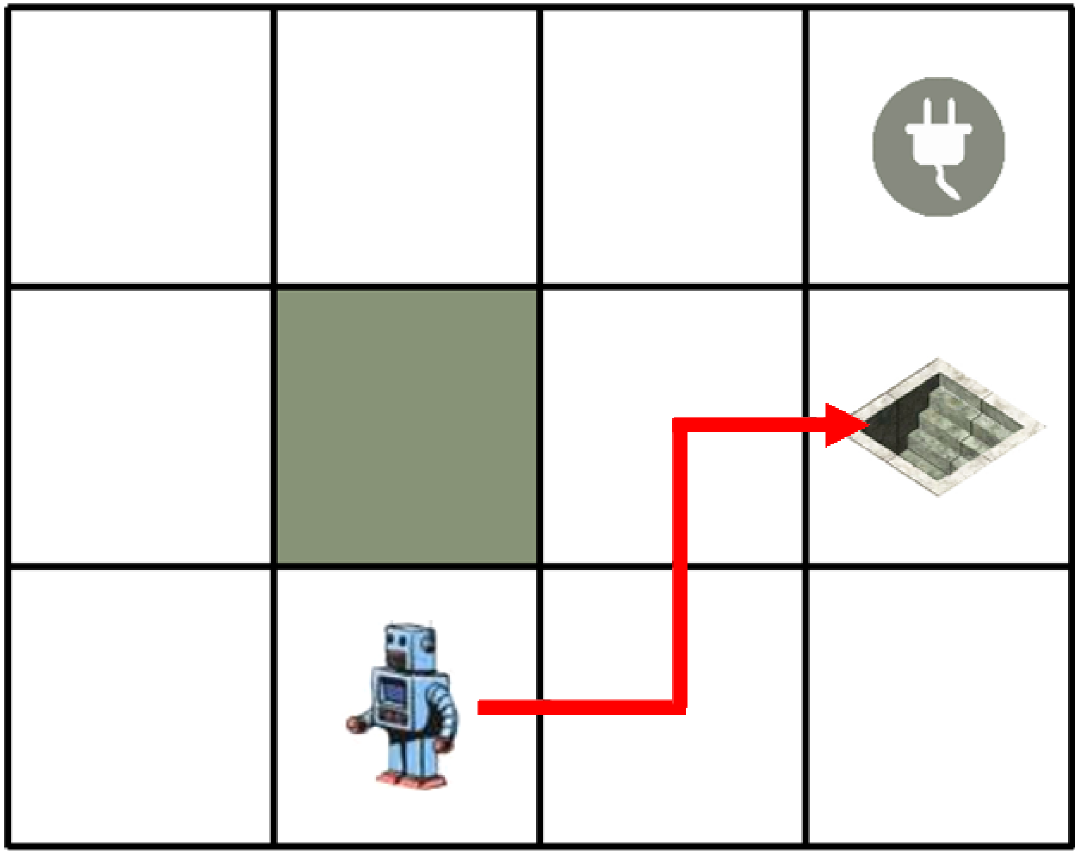
- Executing the **A* plan** in this environment



But: transitions are non-deterministic!

MDP Example

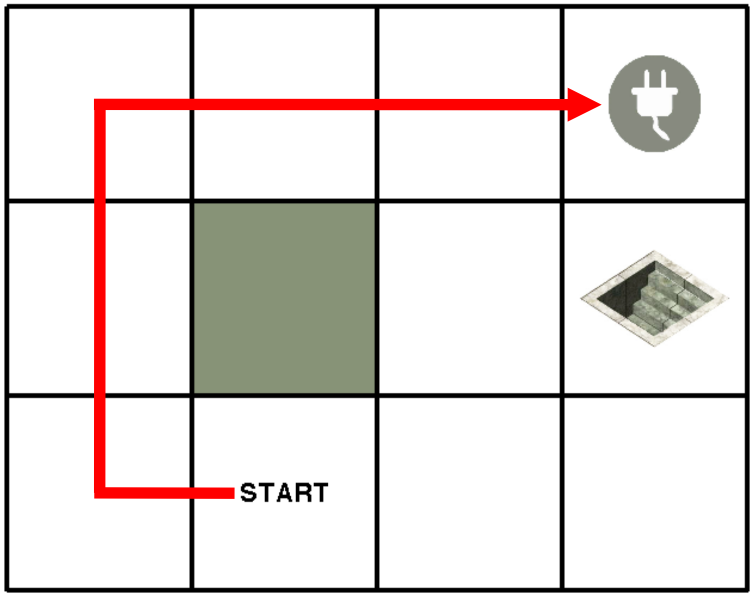
- Executing the **A* plan** in this environment



This will happen sooner or later...

MDP Example

- Use a **longer** path with **lower** probability to end up in cell labelled **-1**



- This path has the **highest overall utility**
- Probability $0.8^6 = 0.2621$

Transition Model

- The probability to reach the next state s' from state s by choosing action a

  $$T(s, a, s')$$

  is called **transition model**

Markov Property:

The transition probabilities from s to s' **depend only on the current state** s and not on the history of earlier states

Reward

- In each state s, the agent receives a **reward** $R(s)$
- The reward may be **positive** or **negative** but must be **bounded**
- This can be generalized to be a function $R(s,a,s')$.
  Here: considering only $R(s)$, does not change the problem

Reward

- In our example, the reward is **-0.04** in all states (e.g. the cost of motion) except the terminal states (that have rewards **+1**/**-1**)
- A negative reward gives agents an **incentive to reach the goal quickly**
- Or: “living in this environment is not enjoyable”

-0.04	-0.04	-0.04	+1
-0.04	▨	-0.04	-1
-0.04	-0.04	-0.04	-0.04

MDP Definition

- Given a **sequential decision problem** in a fully observable, stochastic environment with a known Markovian transition model

- Then a **Markov Decision Process** is defined by the components
  - *Set of states:* S
  - *Set of actions:* A
  - *Initial state:* s_0
  - *Transition model:* $T(s, a, s')$
  - *Reward funciton:* $R(s)$

Policy

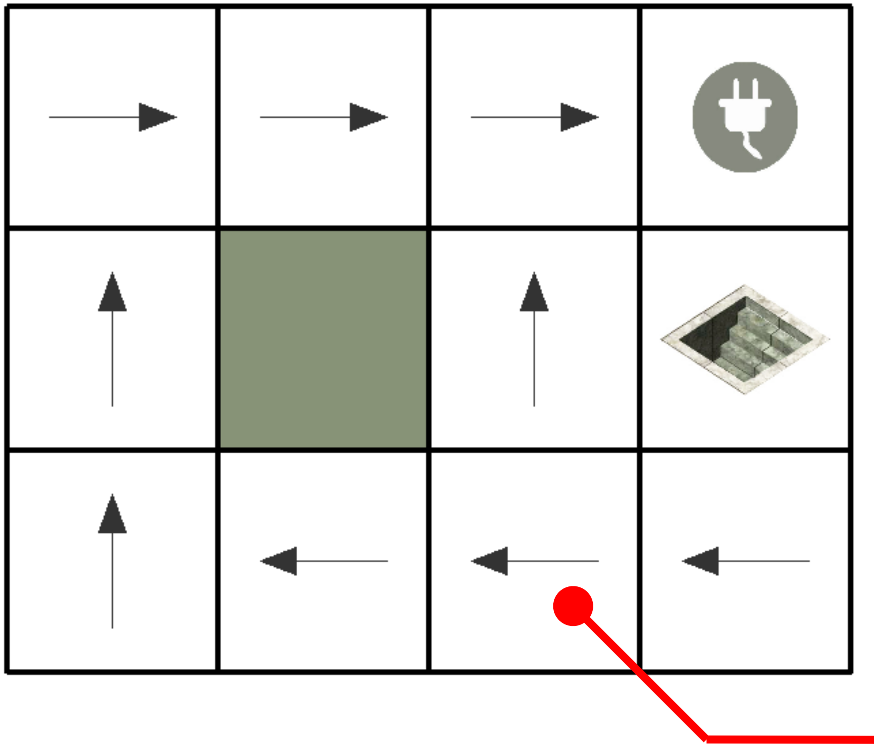
- An MDP solution is called **policy** π
- A policy is a mapping from states to actions

$$policy : States \mapsto Actions$$

- In each state, a policy tells the agent **what to do next**
- Let $\pi(s)$ be the *action* that π specifies for s
- Among the many policies that solve an MDP, the **optimal policy** π* is what we seek. We'll see later what *optimal* means

Policy

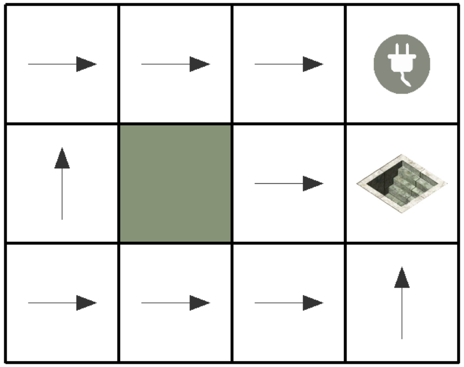
- The optimal policy for our example



Conservative choice
Take long way around as the cost per step of -0.04 is small compared with the penality to fall down the stairs and receive a **-1** reward

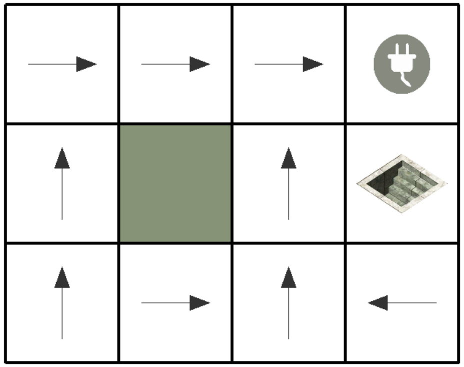
Policy

- When the balance of risk and reward changes, **other policies are optimal**



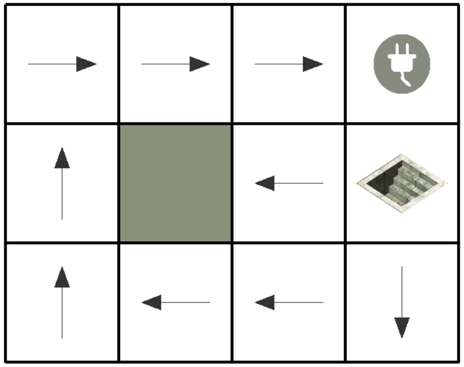
R = -2

Leave as soon as possible



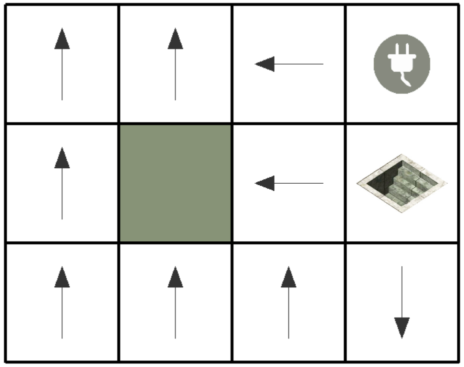
R = -0.2

Take shortcut, minor risks



R = -0.01

No risks are taken



R > 0

Never leave (inf. #policies)

Utility of a State

- The **utility of a state** *U(s)* quantifies the **benefit** of a state for the **overall task**

- We first define $U^{\pi}(s)$ to be the **expected utility of all state sequences that start in** *s* given π

$$U^{\pi}(s) \quad = \quad E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right]$$

- *U(s)* evaluates (and encapsulates) all possible futures **from** *s* **onwards**

Utility of a State

- With this definition, we can express $U^{\pi}(s)$ as a **function of its next state** s'

$$
\begin{aligned}
U^{\pi}(s) &= E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right] \\
&= E\Big[R(s_0) + R(s_1) + R(s_2) + \ldots \mid \pi, s_0 = s\Big] \\
&= E\Big[R(s_0) \mid s_0 = s\Big] + E\Big[R(s_1) + R(s_2) + \ldots \mid \pi\Big] \\
&= R(s) + E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s'\right] \\
&= R(s) + U^{\pi}(s')
\end{aligned}
$$

Optimal Policy

- The utility of a state allows us to apply the **Maximum Expected Utility principle** to define the optimal policy π*

- The **optimal policy** π* in s chooses the action a that maximizes the expected utility of s (and of s')

$$\pi^*(s) \quad = \quad \underset{a}{\operatorname{argmax}} \; E\left[U^\pi(s)\right]$$

- Expectation taken over all policies

Optimal Policy

- Substituting $U^{\pi}(s)$

$$\begin{aligned}
\pi^*(s) &= \operatorname*{argmax}_a E\left[U^{\pi}(s)\right] \\
&= \operatorname*{argmax}_a E\left[R(s) + U^{\pi}(s')\right] \\
&= \operatorname*{argmax}_a E\left[R(s)\right] + E\left[U^{\pi}(s')\right] \\
&= \operatorname*{argmax}_a E\left[U(s')\right] \\
&= \operatorname*{argmax}_a \sum_{s'} T(s, a, s')\, U(s')
\end{aligned}$$

- Recall that $E[X]$ is the weighted average of all possible values that X can take on

Utility of a State

- The **true utility of a state** *U(s)* is then obtained by application of the optimal policy, i.e. $U^{\pi^*}(s) = U(s)$. We find

$$
\begin{aligned}
U(s) &= \max_a E\left[U^\pi(s)\right] \\
&= \max_a E\left[R(s) + U^\pi(s')\right] \\
&= \max_a E\left[R(s)\right] + E\left[U^\pi(s')\right] \\
&= R(s) + \max_a E\left[U(s')\right] \\
&= \underline{R(s) + \max_a \sum_{s'} T(s,a,s')\, U(s')}
\end{aligned}
$$

Utility of a State

- This result is noteworthy:

$$U(s) = R(s) + \max_a \sum_{s'} T(s, a, s')\, U(s')$$

  We have found a direct relationship between the **utility of a state** and the **utility of its neighbors**

- The utility of a state is the immediate reward for that state plus the expected utility of the next state, **provided** the agent chooses the **optimal** action

Bellman Equation

$$\boxed{U(s) = R(s) + \max_{a} \sum_{s'} T(s, a, s')\, U(s')}$$

- For each state there is a Bellman equation to compute its utility
- There are ***n* states** and ***n* unknowns**
- Solve the system using Linear Algebra?
- No! The max-operator that chooses the optimal action makes the system nonlinear
- We must go for an **iterative approach**

Discounting

We have made a **simplification** on the way:

- The utility of a state sequence is often defined as the sum of **discounted** rewards

  $$U^\pi(s) = E\left[\sum_{t=0}^{\infty} \underline{\gamma^t} R(s_t) \mid \pi, s_0 = s\right]$$

  with $0 \, \delta \, \gamma \, \delta \, 1$ being the *discount factor*
- Discounting says that **future** rewards are **less significant** than **current** rewards. This is a natural model for many domains
- The other expressions change accordingly

Separability

We have made an **assumption** on the way:

- Not all utility functions (for state sequences) can be used
- The utility function must have the **property of separability** (a.k.a. stationarity), e.g. additive utility functions:

$$U([s_0 + s_1 + \ldots + s_n]) = R(s_0) + U([s_1 + \ldots + s_n])$$

- Loosely speaking: the preference between two state sequences is unchanged over different start states

Utility of a State

- The **state utilities** for our example

0.812	0.868	0.918	+1
0.762		0.66	-1
0.705	0.655	0.611	0.388

- Note that utilities are higher closer to the goal as fewer steps are needed to reach it

Iterative Computation

Idea:

- The utility is computed iteratively:

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s,a,s')\, U_i(s')$$

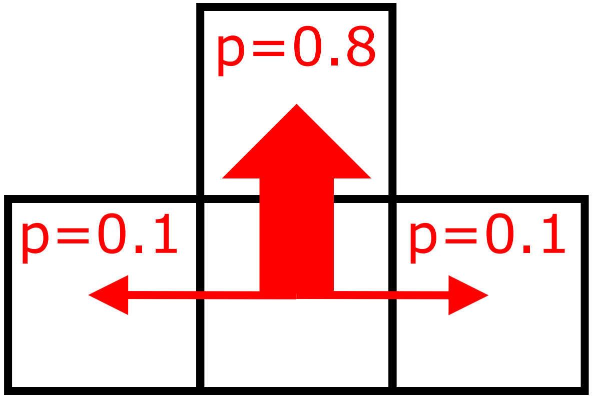
- Optimal utility: $U^* = \lim_{t \to \infty} U_t$
- Abort, if change in utility is below a threshold

Value Iteration Example

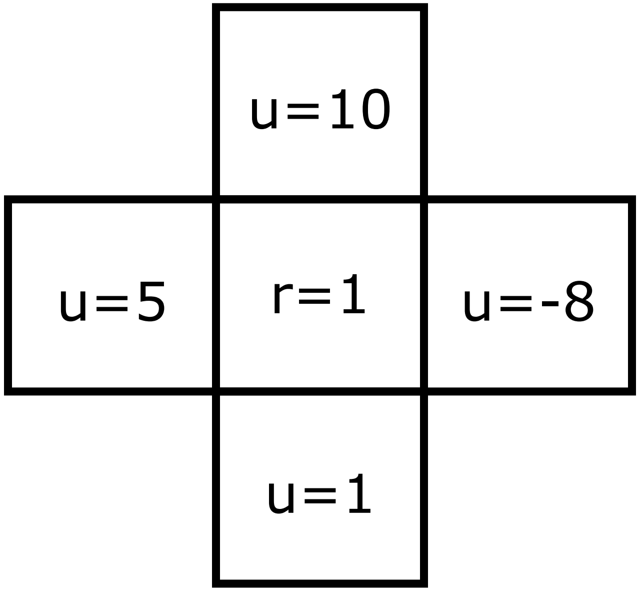
- Calculate utility of the center cell

$$U_{i+1}(s) \leftarrow R(s) + \max_{a} \sum_{s'} T(s,a,s')\, U_i(s')$$

desired action = *Up*



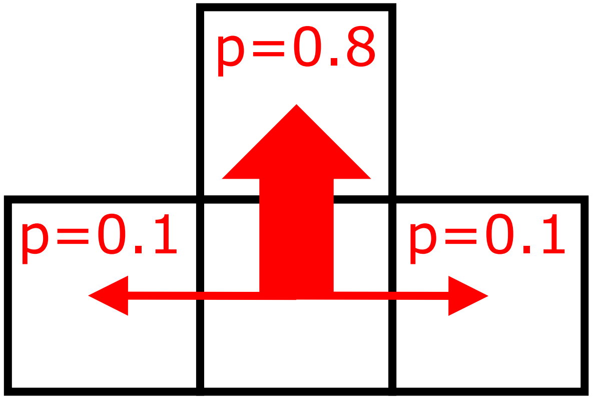
Transition Model

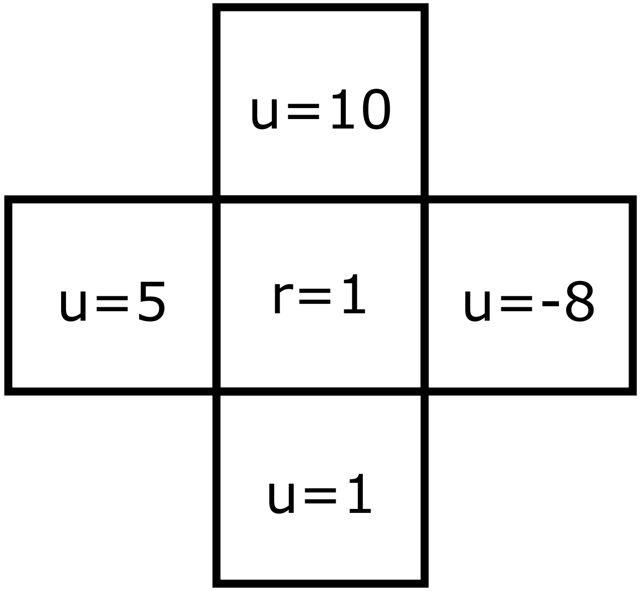


State space
(u=utility, r=reward)

Value Iteration Example

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s,a,s')\, U_i(s')$$

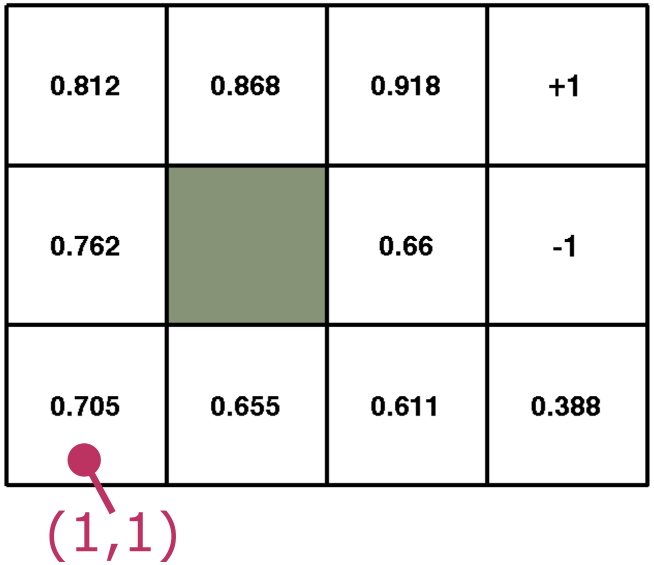




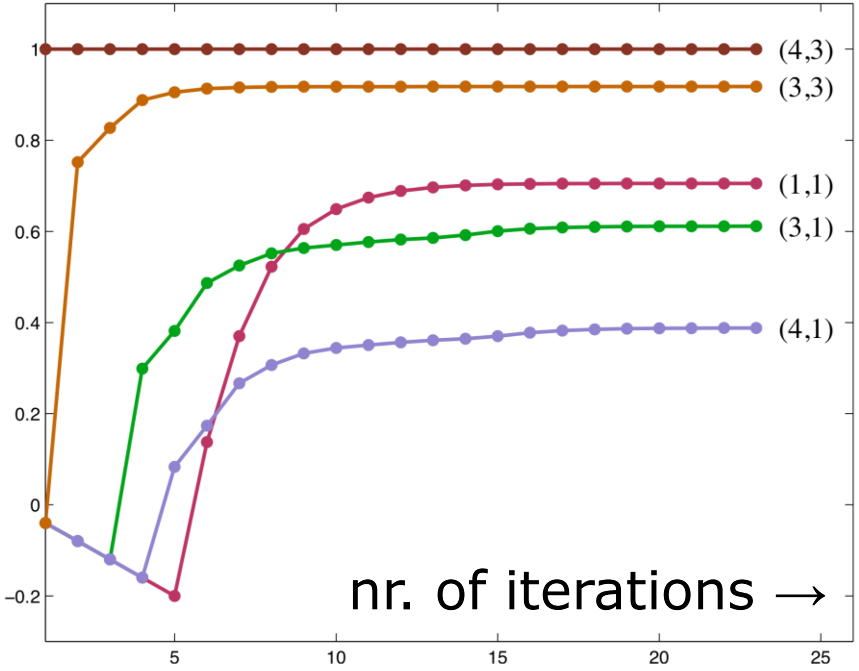
$$\begin{aligned}
&= \mathit{reward} + \text{max}\{ \\
&\quad 0.1 \cdot 1 + 0.8 \cdot 5 + 0.1 \cdot 10 \quad (\leftarrow), \\
&\quad 0.1 \cdot 5 + 0.8 \cdot 10 + 0.1 \cdot -8 \quad (\uparrow), \\
&\quad 0.1 \cdot 10 + 0.8 \cdot -8 + 0.1 \cdot 1 \quad (\rightarrow), \\
&\quad 0.1 \cdot -8 + 0.8 \cdot 1 + 0.1 \cdot 5 \quad (\downarrow)\} \\
&= 1 + \text{max}\{5.1\,(\leftarrow), 7.7\,(\uparrow), \\
&\quad -5.3\,(\rightarrow), 0.5\,(\downarrow)\} \\
&= 1 + 7.7 \\
&= 8.7
\end{aligned}$$

Value Iteration Example

- In our example



0.812	0.868	0.918	+1
0.762		0.66	-1
0.705	0.655	0.611	0.388



- States far from the goal first accumulate negative rewards until a path is found to the goal

Convergence

- The condition $close\text{-}enough(U, U')$ in the algorithm can be formulated by

$$RMS = \frac{1}{|S|}\sqrt{\sum_s (U(s) - U'(s))^2}$$

$$RMS(U, U') < \epsilon$$

- Different ways to detect convergence:
  - RMS error: root mean square error
  - Max error: $\| U - U' \| = \max_s |U(s) - U'(s)|$
  - Policy loss

Value Iteration

- Value Iteration finds the **optimal solution** to the Markov Decision Problem!
- **Converges** to the **unique solution** of the Bellman equation system
- Initial values for U' are arbitrary
- Proof involves the concept of *contraction*. $\| B\, U_i - B\, U_i' \| \;\leq\; \gamma \, \| U_i - U_i' \|$ with B being the Bellman operator (see textbook)
- VI propagates information through the state space by means of **local updates**

Optimal Policy

- How to finally compute the **optimal policy**? Can be easily extracted along the way by

$$\pi^*(s) = \operatorname*{argmax}_a \sum_{s'} T(s, a, s') \, U(s')$$

- **Note:** *U(s)* and *R(s)* are quite different quantities. *R(s)* is the **short-term** reward for being in *s*, whereas *U(s)* is the **long-term** reward **from** *s* **onwards**

Summary

- Robust navigation requires combined path planning & collision avoidance.
- Approaches need to consider robot's kinematic constraints and plans in the velocity space.
- Combination of search and reactive techniques show better results than the pure DWA in a variety of situations.
- Using the 5D-approach the quality of the trajectory scales with the performance of the underlying hardware.
- The resulting paths are often close to the optimal ones.

Summary

- Planning is a complex problem.
- Focus on subset of the configuration space:
  - road maps,
  - grids.
- Sampling algorithms are faster and have a trade-off between optimality and speed.
- Uncertainty in motion leads to the need of Markov Decision Problems.

What’s Missing?

- More complex vehicles (e.g., cars, legged robots, manipulators, …).
- Moving obstacles, motion prediction.
- High dimensional spaces.
- Heuristics for improved performances.
- Learning.