

Introduction to Mobile Robotics

Probabilistic Motion Models

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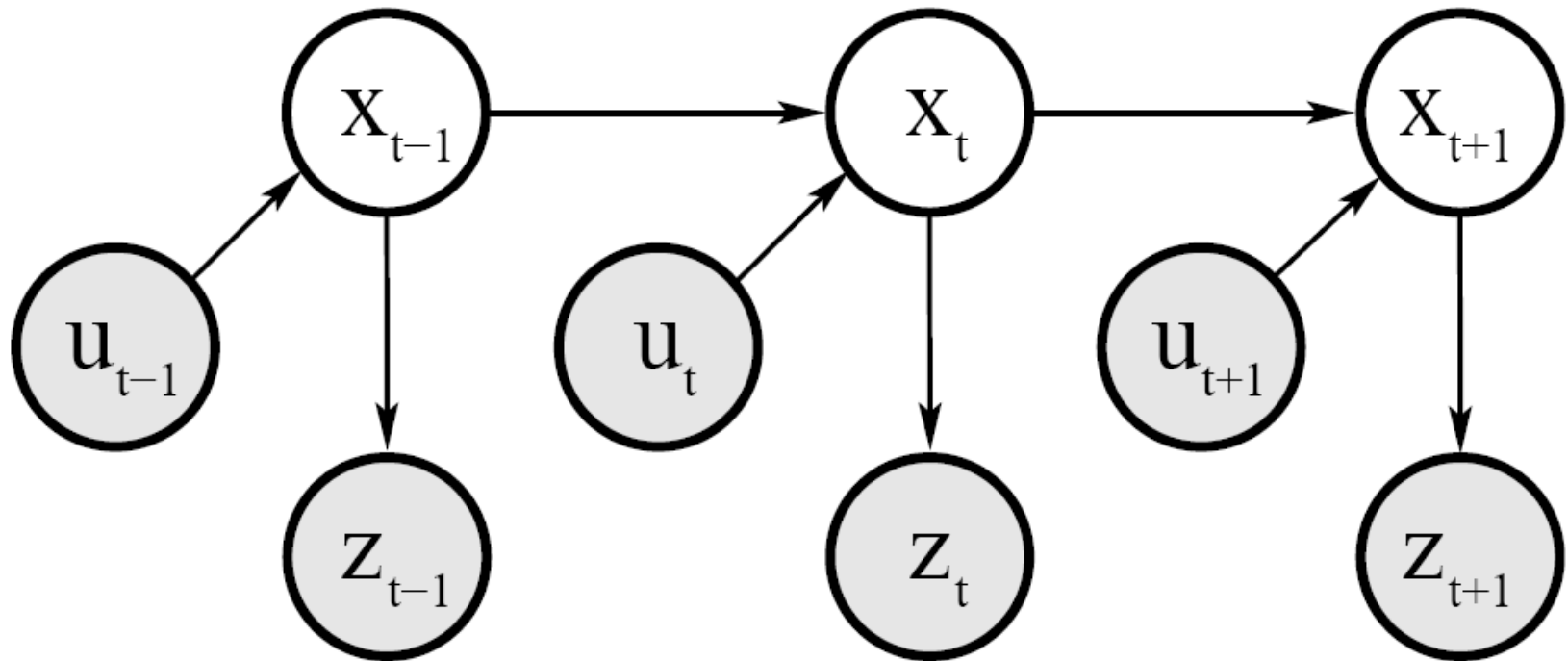


Robot Motion

- Robot motion is inherently uncertain.
- How can we model this uncertainty?



Dynamic Bayesian Network for Controls, States, and Sensations

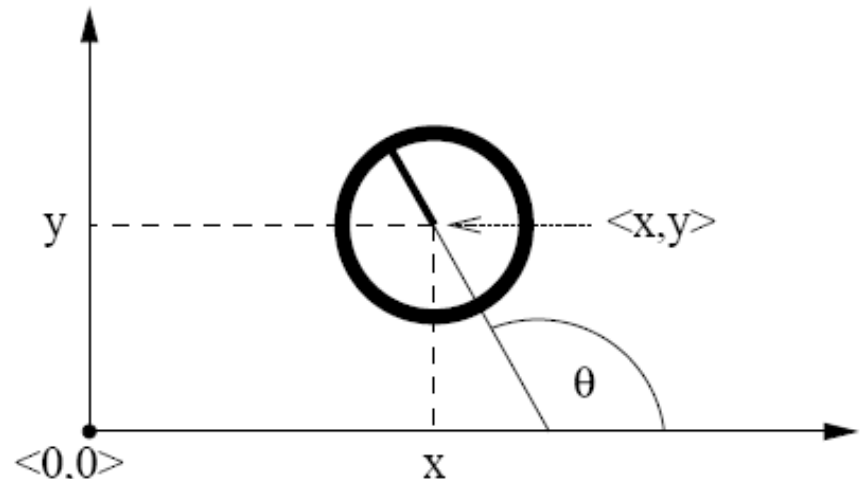


Probabilistic Motion Models

- To implement the Bayes Filter, we need the transition model $p(x_t | x_{t-1}, u_t)$.
- The term $p(x_t | x_{t-1}, u_t)$ specifies a posterior probability, that action u_t carries the robot from x_{t-1} to x_t .
- In this section we will discuss, how $p(x_t | x_{t-1}, u_t)$ can be modeled based on the motion equations and the uncertain outcome of the movements.

Coordinate Systems

- The configuration of a typical wheeled robot in 3D can be described by six parameters.
- These are the three-dimensional Cartesian coordinates plus the three Euler angles for roll, pitch, and yaw.
- For simplicity, throughout this section we consider robots operating on a planar surface.
- The state space of such systems is three-dimensional (x, y, θ) .

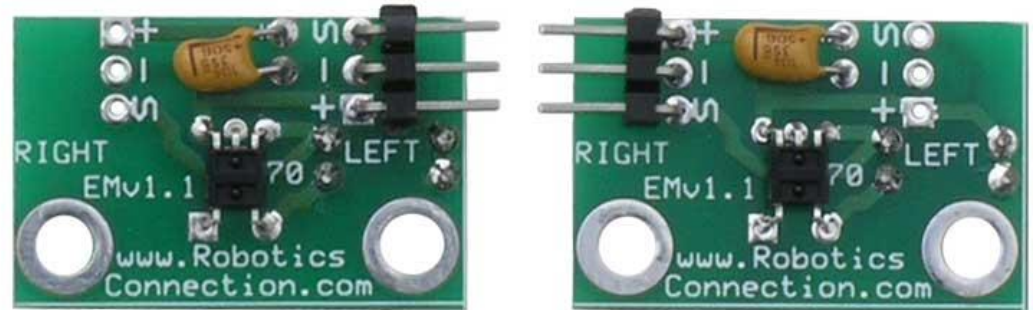


Typical Motion Models

- In practice, one often finds two types of motion models:
 - **Odometry-based**
 - **Velocity-based (dead reckoning)**
- Odometry-based models are used when systems are equipped with wheel encoders.
- Velocity-based models have to be applied when no wheel encoders are given.
- They calculate the new pose based on the velocities and the time elapsed.

Example Wheel Encoders

These modules provide +5V output when they "see" white, and a 0V output when they "see" black.



These disks are manufactured out of high quality laminated color plastic to offer a very crisp black to white transition. This enables a wheel encoder sensor to easily see the transitions.

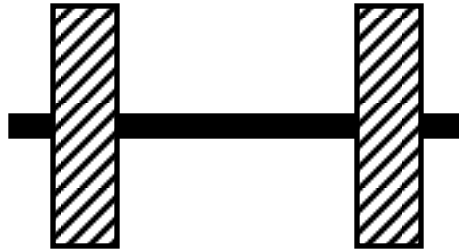
Dead Reckoning

- Derived from “deduced reckoning.”
- Mathematical procedure for determining the present location of a vehicle.
- Achieved by calculating the current pose of the vehicle based on its velocities and the time elapsed.
- Historically used to log the position of ships.

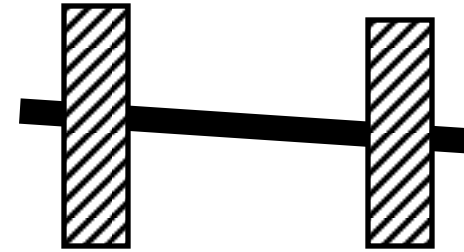


[Image source:
Wikipedia, LoKiLeCh] 8

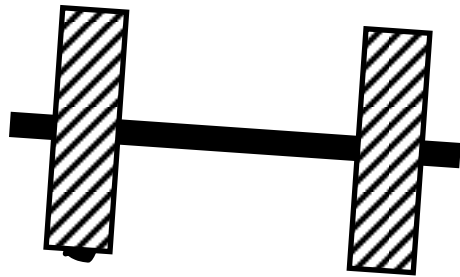
Reasons for Motion Errors of Wheeled Robots



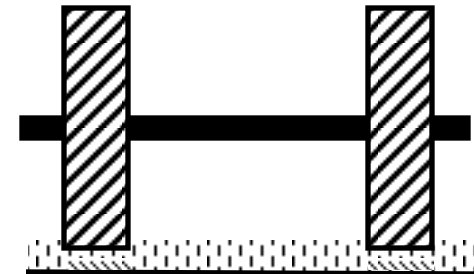
ideal case



different wheel diameters



bump



carpet

and many more ...

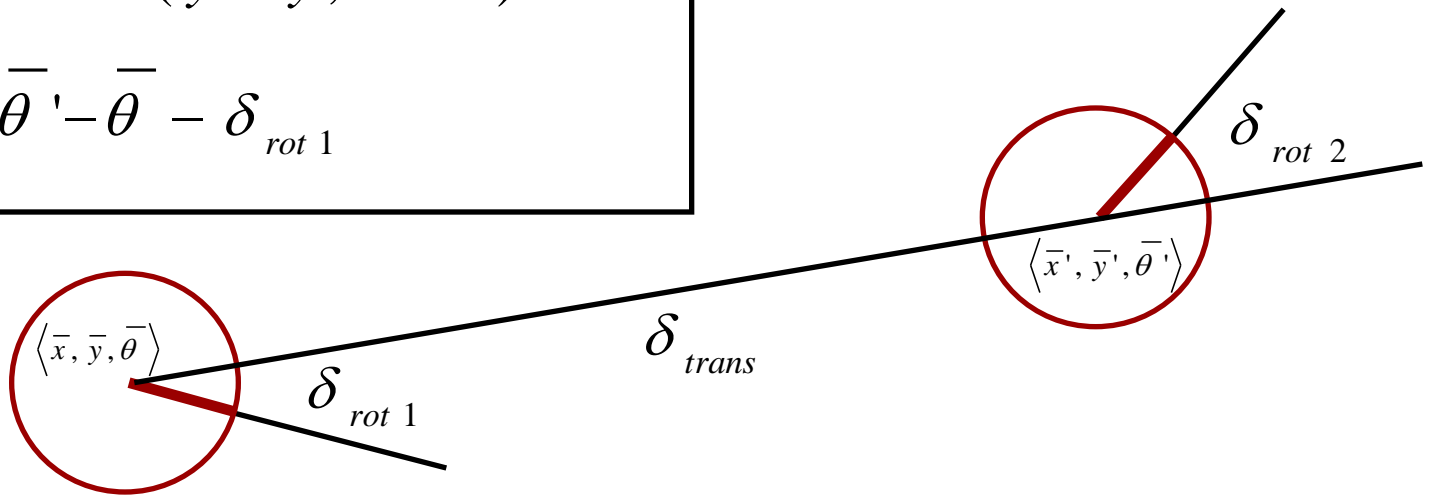
Odometry Model

- Robot moves from $\langle \bar{x}, \bar{y}, \bar{\theta} \rangle$ to $\langle \bar{x}', \bar{y}', \bar{\theta}' \rangle$.
- Odometry information $u = \langle \delta_{rot\ 1}, \delta_{rot\ 2}, \delta_{trans} \rangle$.

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot\ 1} = \text{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

$$\delta_{rot\ 2} = \bar{\theta}' - \bar{\theta} - \delta_{rot\ 1}$$



The atan2 Function

- Extends the inverse tangent and correctly copes with the signs of x and y .

$$\text{atan2}(y, x) = \begin{cases} \text{atan}(y/x) & \text{if } x > 0 \\ \text{sign}(y) (\pi - \text{atan}(|y/x|)) & \text{if } x < 0 \\ 0 & \text{if } x = y = 0 \\ \text{sign}(y) \pi/2 & \text{if } x = 0, y \neq 0 \end{cases}$$

Noise Model for Odometry

- The measured motion is given by the true motion corrupted with noise.

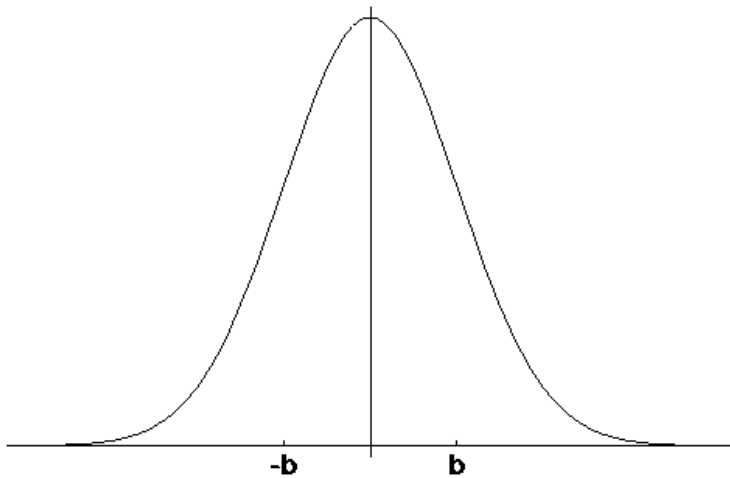
$$\hat{\delta}_{rot\ 1} = \delta_{rot\ 1} + \epsilon_{\alpha_1 |\delta_{rot\ 1}| + \alpha_2 |\delta_{trans}|}$$

$$\hat{\delta}_{trans} = \delta_{trans} + \epsilon_{\alpha_3 |\delta_{trans}| + \alpha_4 (|\delta_{rot\ 1}| + |\delta_{rot\ 2}|)}$$

$$\hat{\delta}_{rot\ 2} = \delta_{rot\ 2} + \epsilon_{\alpha_1 |\delta_{rot\ 2}| + \alpha_2 |\delta_{trans}|}$$

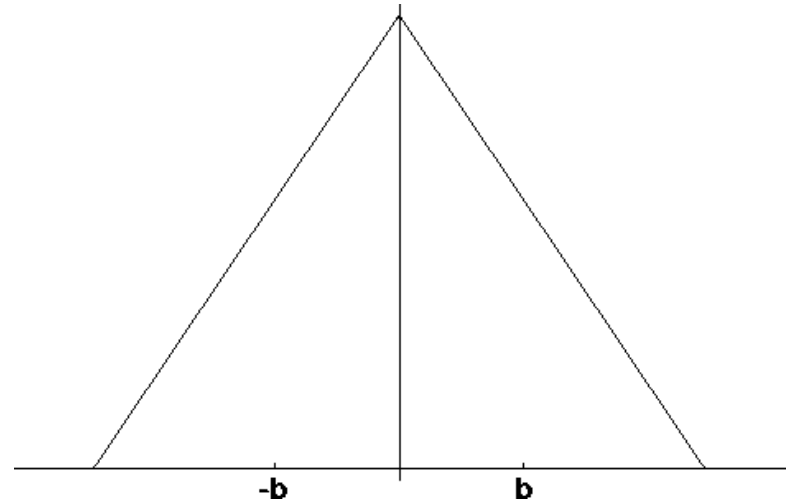
Typical Distributions for Probabilistic Motion Models

Normal distribution



$$\varepsilon_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

Triangular distribution



$$\varepsilon_{\sigma^2}(x) = \begin{cases} 0 & \text{if } |x| > \sqrt{6\sigma^2} \\ \frac{\sqrt{6\sigma^2} - |x|}{6\sigma^2} & \text{otherwise} \end{cases}$$

Calculating the Probability Density (zero-centered)

- For a normal distribution

1. Algorithm **prob_normal_distribution**(a, b):

query point

std. deviation

2. return $\frac{1}{\sqrt{2\pi} b^2} \exp\left\{-\frac{1}{2} \frac{a^2}{b^2}\right\}$

- For a triangular distribution

1. Algorithm **prob_triangular_distribution**(a, b):

2. return $\max\left\{0, \frac{1}{\sqrt{6} b} - \frac{|a|}{6 b^2}\right\}$

Calculating the Posterior Given \mathbf{x}, \mathbf{x}' , and Odometry

hypotheses odometry

1. Algorithm **motion_model_odometry**(\mathbf{x}, \mathbf{x}') ($\bar{\mathbf{x}}, \bar{\mathbf{x}'}$)

2. $\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$

3. $\delta_{rot1} = \text{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$

4. $\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$

5. $\hat{\delta}_{trans} = \sqrt{(x' - x)^2 + (y' - y)^2}$

6. $\hat{\delta}_{rot1} = \text{atan2}(y' - y, x' - x) - \theta$

7. $\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$

8. $p_1 = \text{prob}(\delta_{rot1} - \hat{\delta}_{rot1}, \alpha_1 \mid |\delta_{rot1}| + \alpha_2 \delta_{trans})$

9. $p_2 = \text{prob}(\delta_{trans} - \hat{\delta}_{trans}, \alpha_3 \delta_{trans} + \alpha_4 (|\delta_{rot1}| + |\delta_{rot2}|))$

10. $p_3 = \text{prob}(\delta_{rot2} - \hat{\delta}_{rot2}, \alpha_1 \mid |\delta_{rot2}| + \alpha_2 \delta_{trans})$

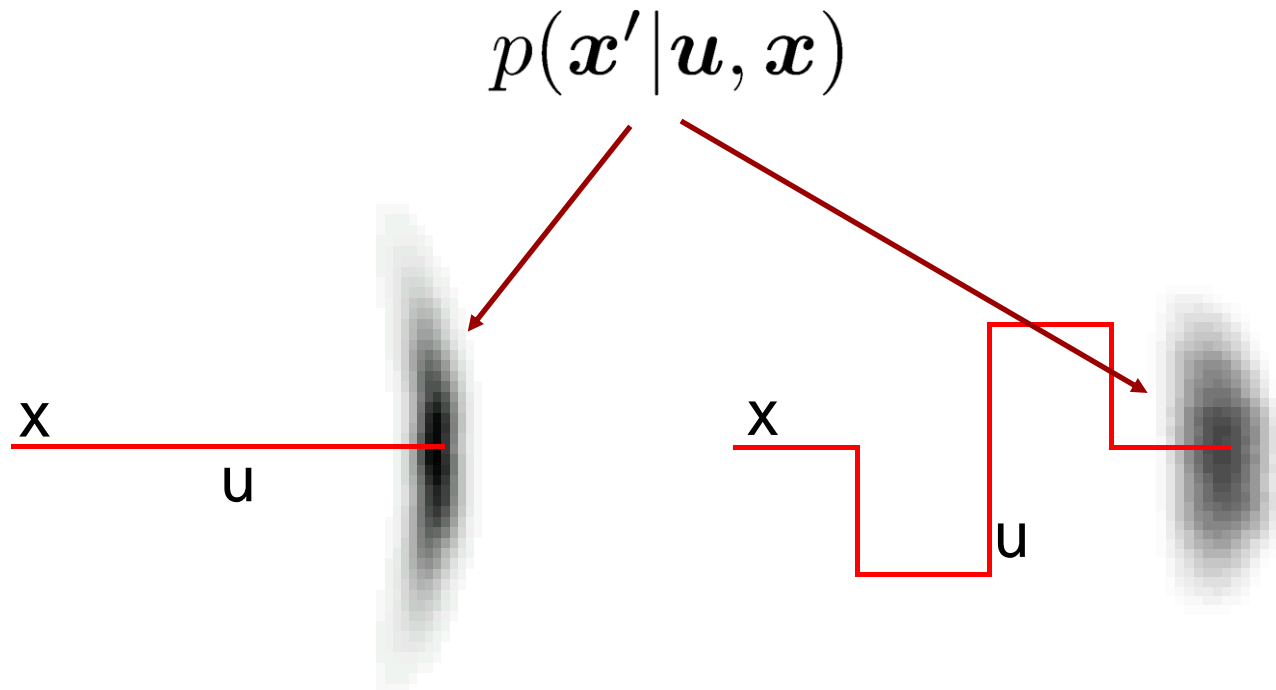
11. return $p_1 \cdot p_2 \cdot p_3$

odometry params (\mathbf{u})

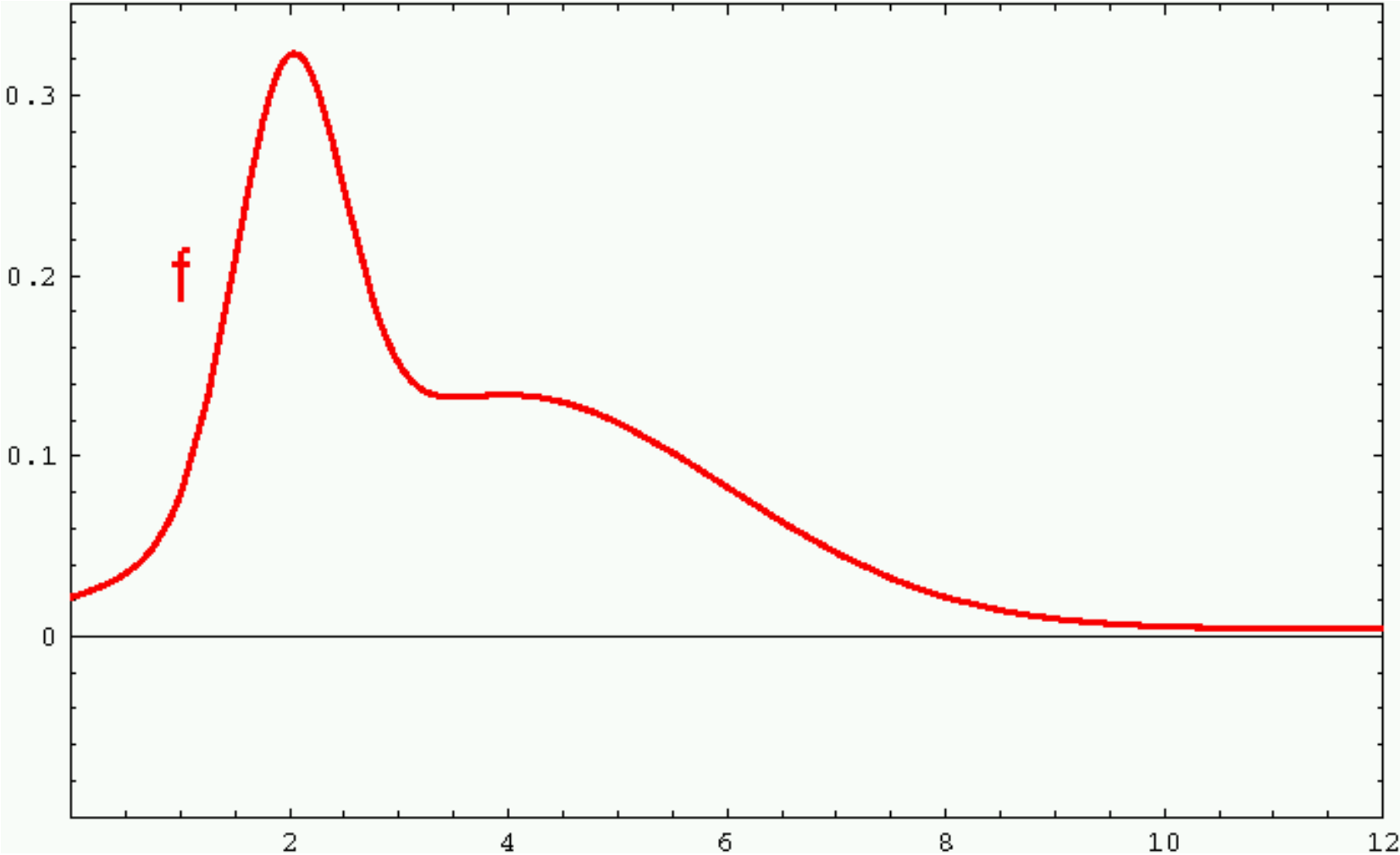
values of interest (\mathbf{x}, \mathbf{x}')

Application

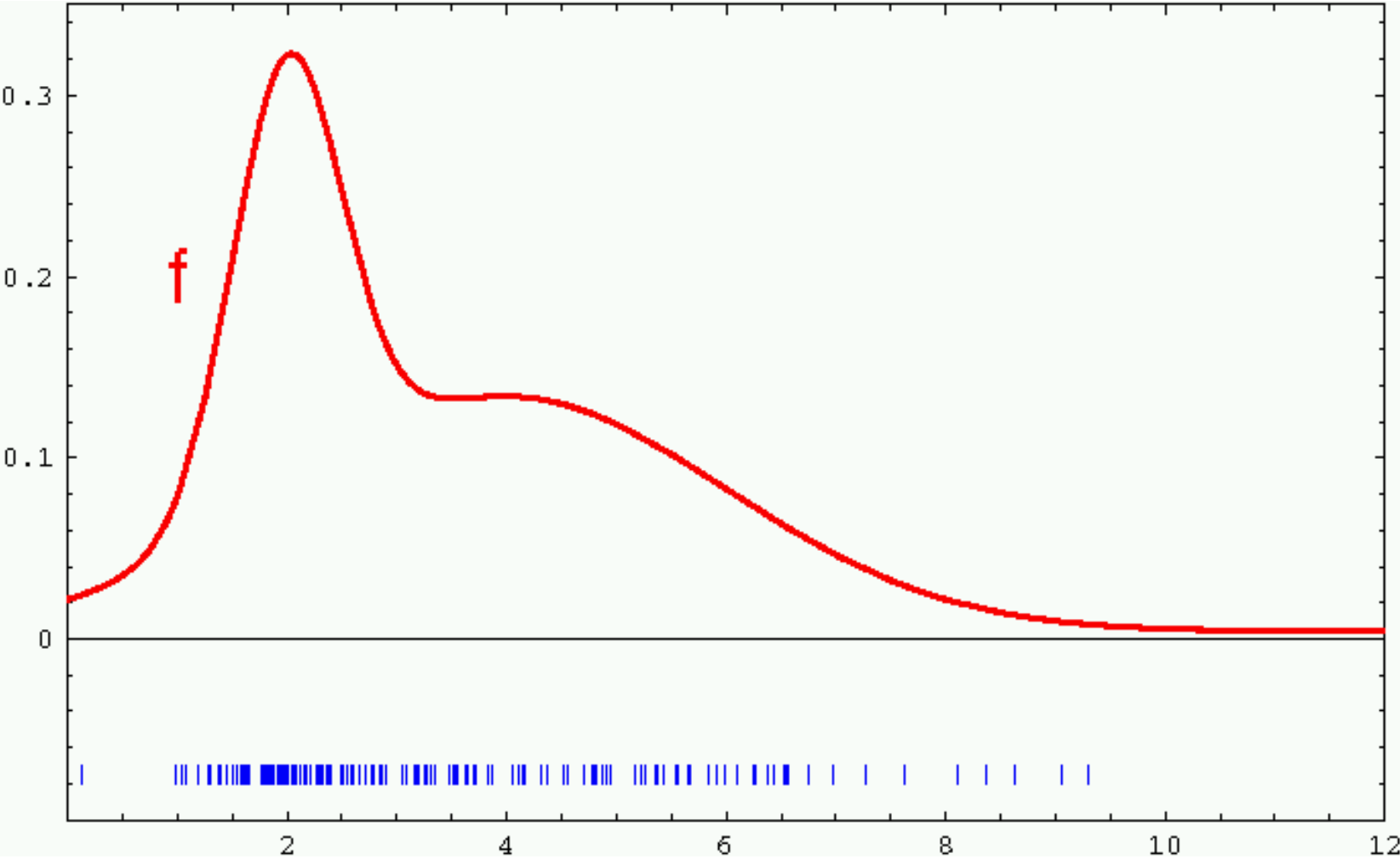
- Repeated application of the motion model for short movements.
- Typical banana-shaped distributions obtained for the 2d-projection of the 3d posterior.



Sample-Based Density Representation



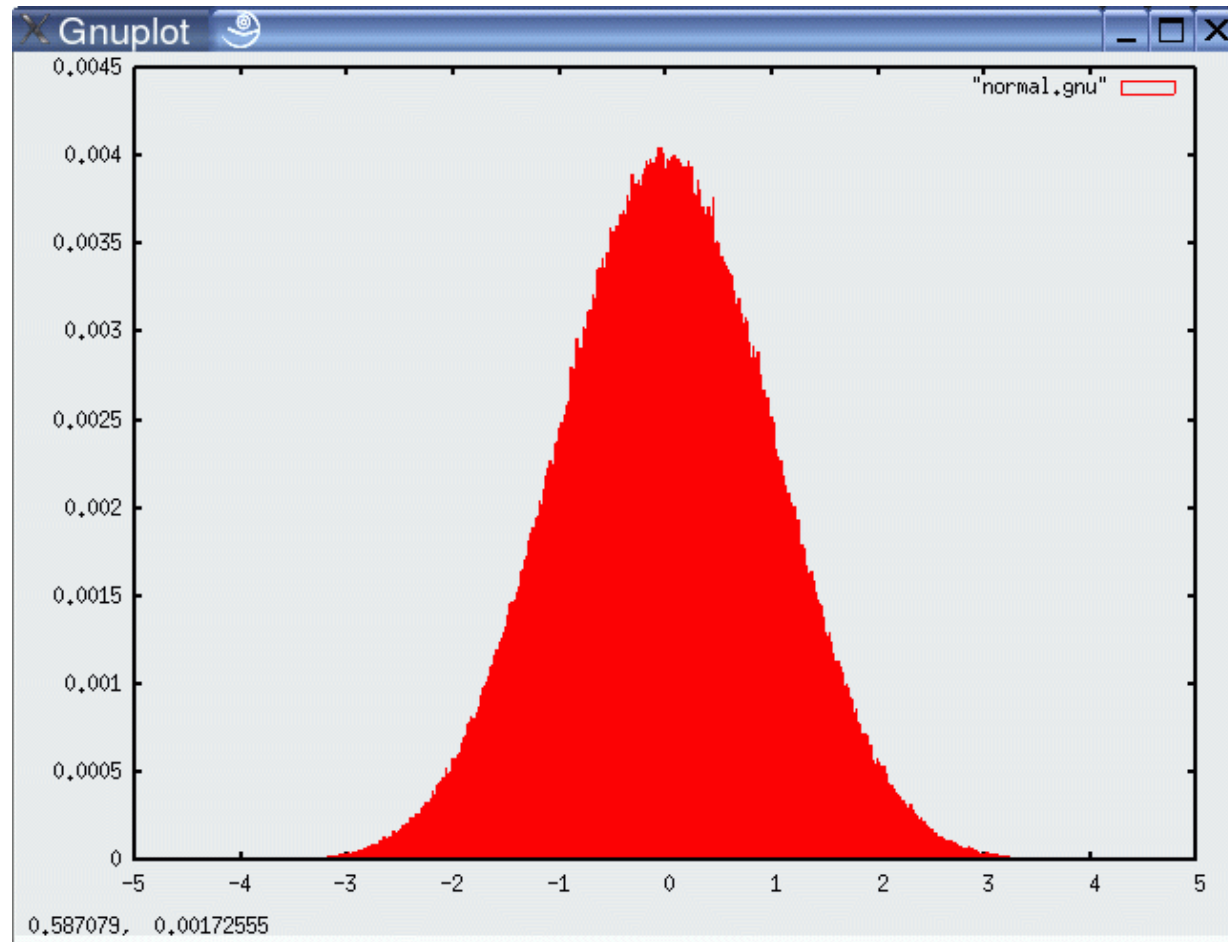
Sample-Based Density Representation



How to Sample from a Normal Distribution?

- Sampling from a normal distribution
 1. Algorithm **sample_normal_distribution**(b):
 2. return $\frac{1}{2} \sum_{i=1}^{12} rand(-b, b)$

Normally Distributed Samples



10⁶ samples

How to Sample from Normal or Triangular Distributions?

- Sampling from a normal distribution

1. Algorithm **sample_normal_distribution**(b):

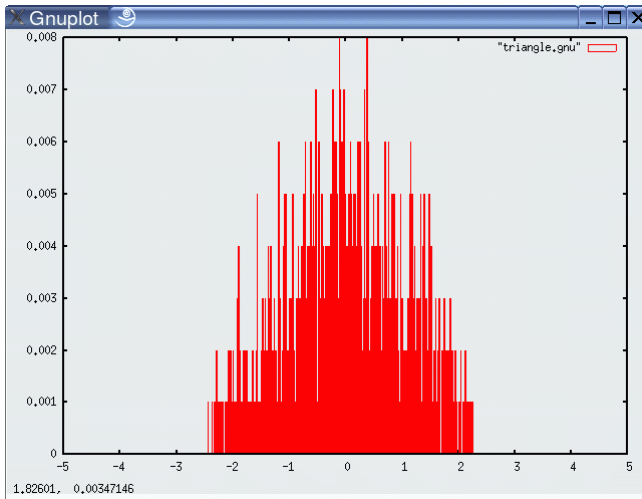
2. return $\frac{1}{2} \sum_{i=1}^{12} \text{rand}(-b, b)$

- Sampling from a triangular distribution

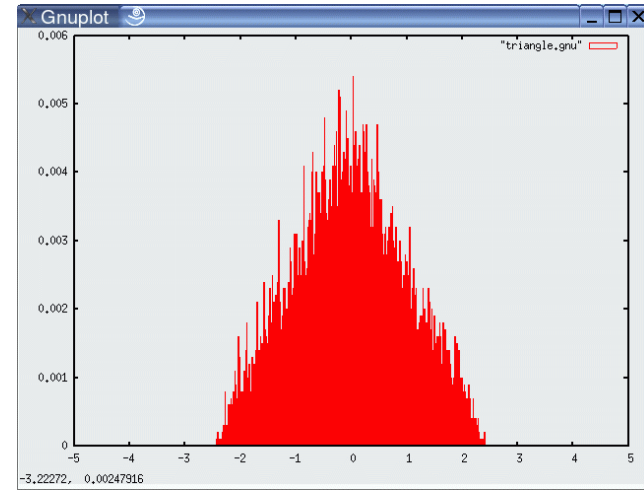
1. Algorithm **sample_triangular_distribution**(b):

2. return $\frac{\sqrt{6}}{2} [\text{rand}(-b, b) + \text{rand}(-b, b)]$

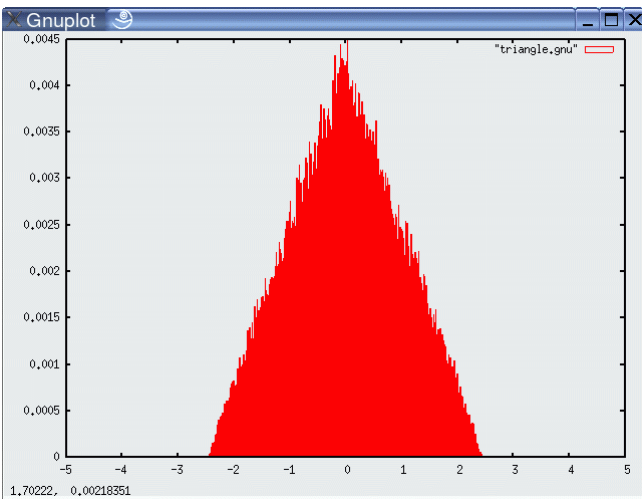
For Triangular Distribution



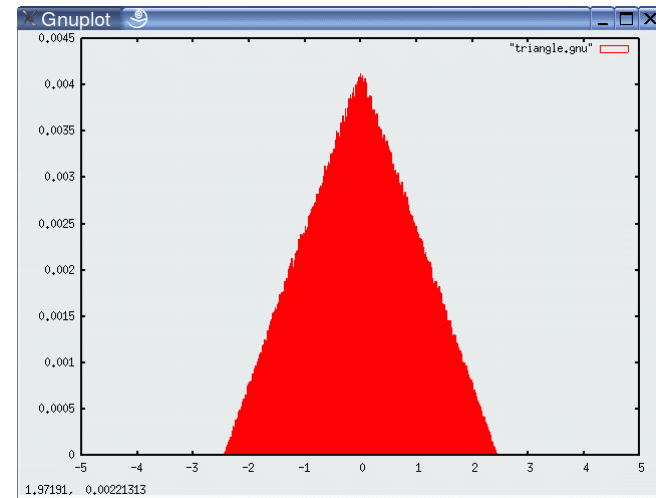
10^3 samples



10^4 samples

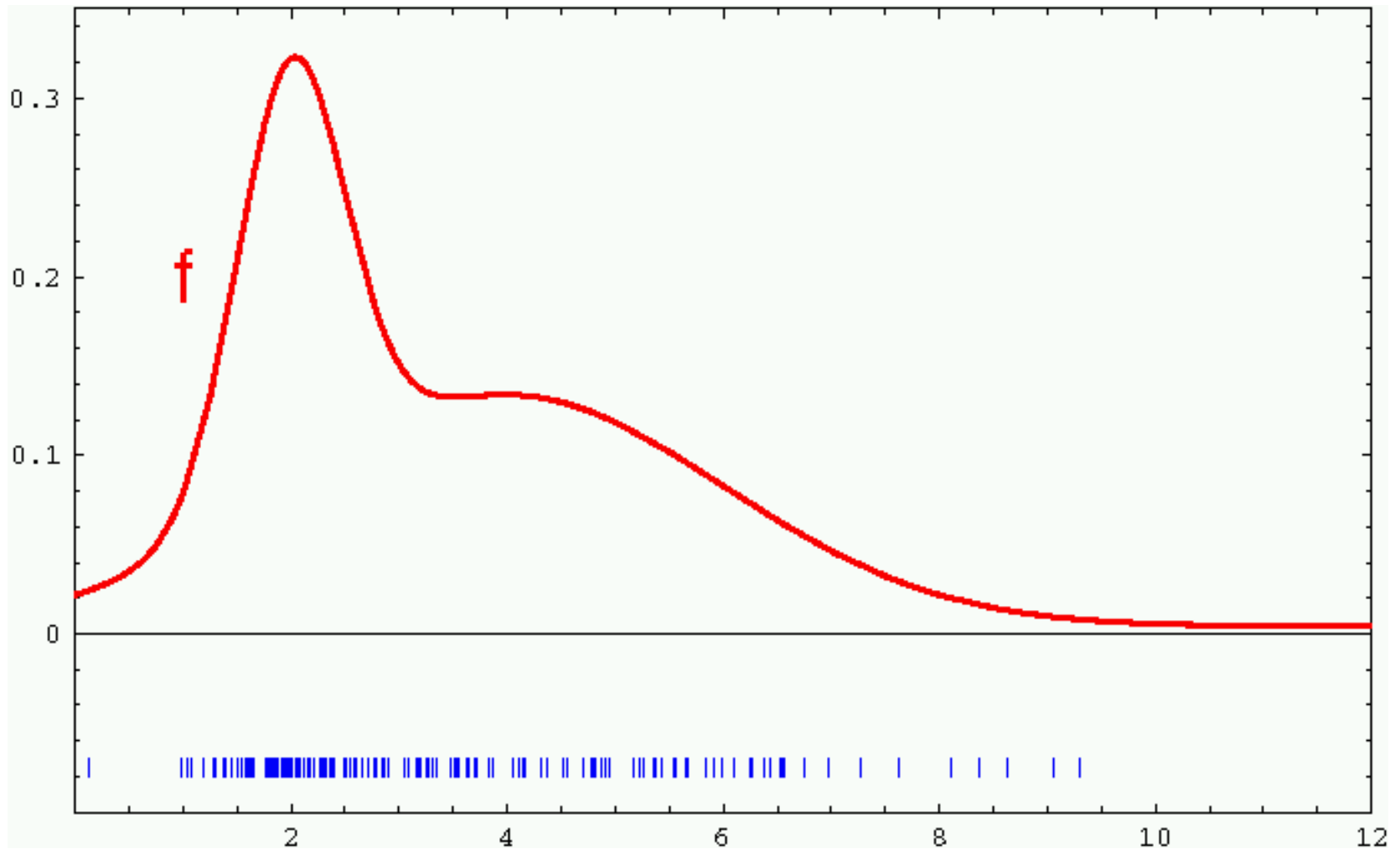


10^5 samples



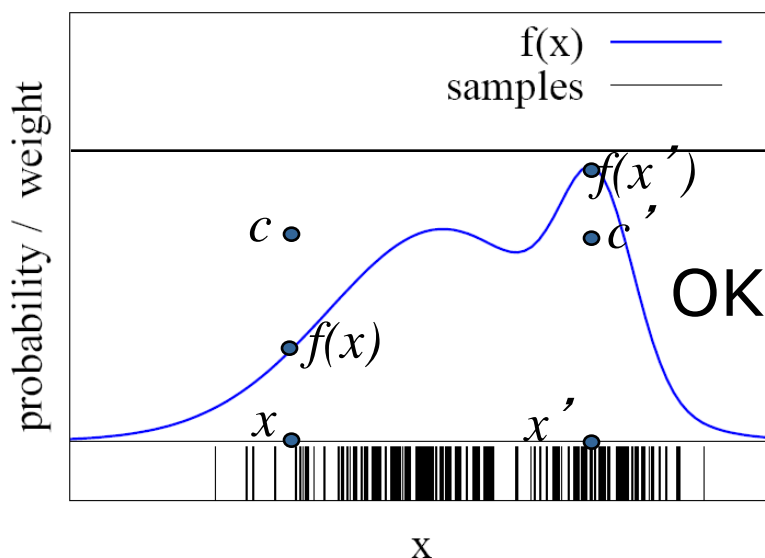
10^6 samples

How to Obtain Samples from Arbitrary Functions?



Rejection Sampling

- Sampling from arbitrary distributions
- Sample x from a uniform distribution from $[-b,b]$
- Sample c from $[0, \max f]$
- if $f(x) > c$ keep the sample
otherwise reject the sample



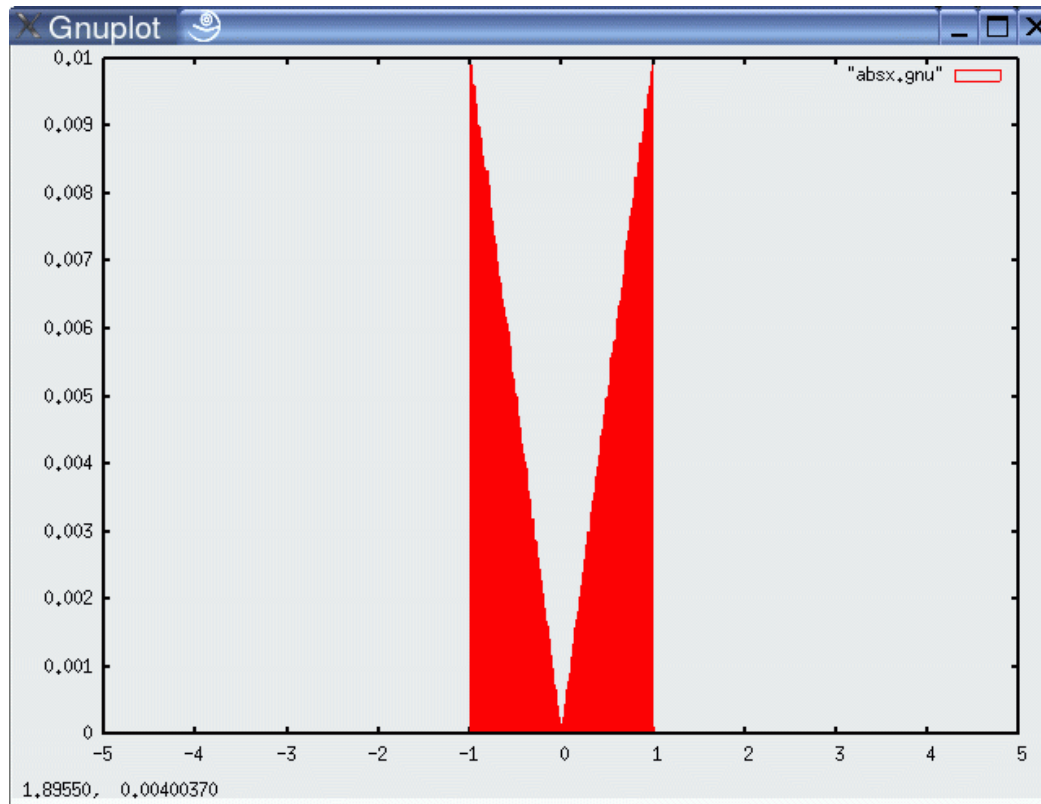
Rejection Sampling

- Sampling from arbitrary distributions
 1. Algorithm **sample_distribution**(f, b):
 2. repeat
 3. $x = \text{rand}(-b, b)$
 4. $y = \text{rand}(0, \max\{f(x) \mid x \in [-b, b]\})$
 5. until ($y \leq f(x)$)
 6. return x

Example

- Sampling from

$$f(x) = \begin{cases} \text{abs}(x) & x \in [-1; 1] \\ 0 & \text{otherwise} \end{cases}$$



Sample Odometry Motion Model

1. Algorithm **sample_motion_model**(u, x):

$$u = \langle \delta_{rot\ 1}, \delta_{rot\ 2}, \delta_{trans} \rangle, x = \langle x, y, \theta \rangle$$

1. $\hat{\delta}_{rot\ 1} = \delta_{rot\ 1} + \text{sample}(\alpha_1 |\delta_{rot\ 1}| + \alpha_2 \delta_{trans})$

2. $\hat{\delta}_{trans} = \delta_{trans} + \text{sample}(\alpha_3 \delta_{trans} + \alpha_4 (|\delta_{rot\ 1}| + |\delta_{rot\ 2}|))$

3. $\hat{\delta}_{rot\ 2} = \delta_{rot\ 2} + \text{sample}(\alpha_1 |\delta_{rot\ 2}| + \alpha_2 \delta_{trans})$

4. $x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot\ 1})$

5. $y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot\ 1})$

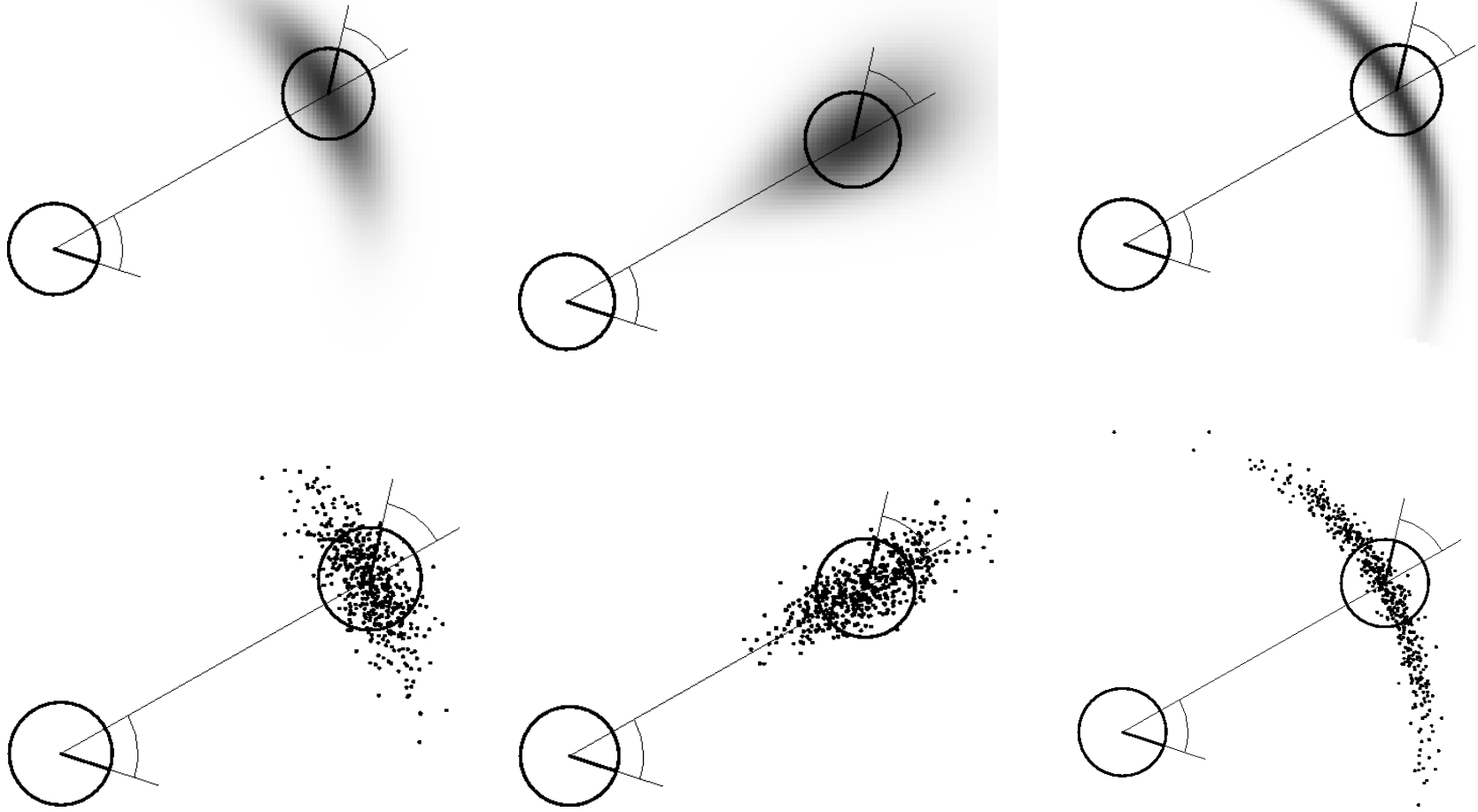
6. $\theta' = \theta + \hat{\delta}_{rot\ 1} + \hat{\delta}_{rot\ 2}$

7. Return $\langle x', y', \theta' \rangle$

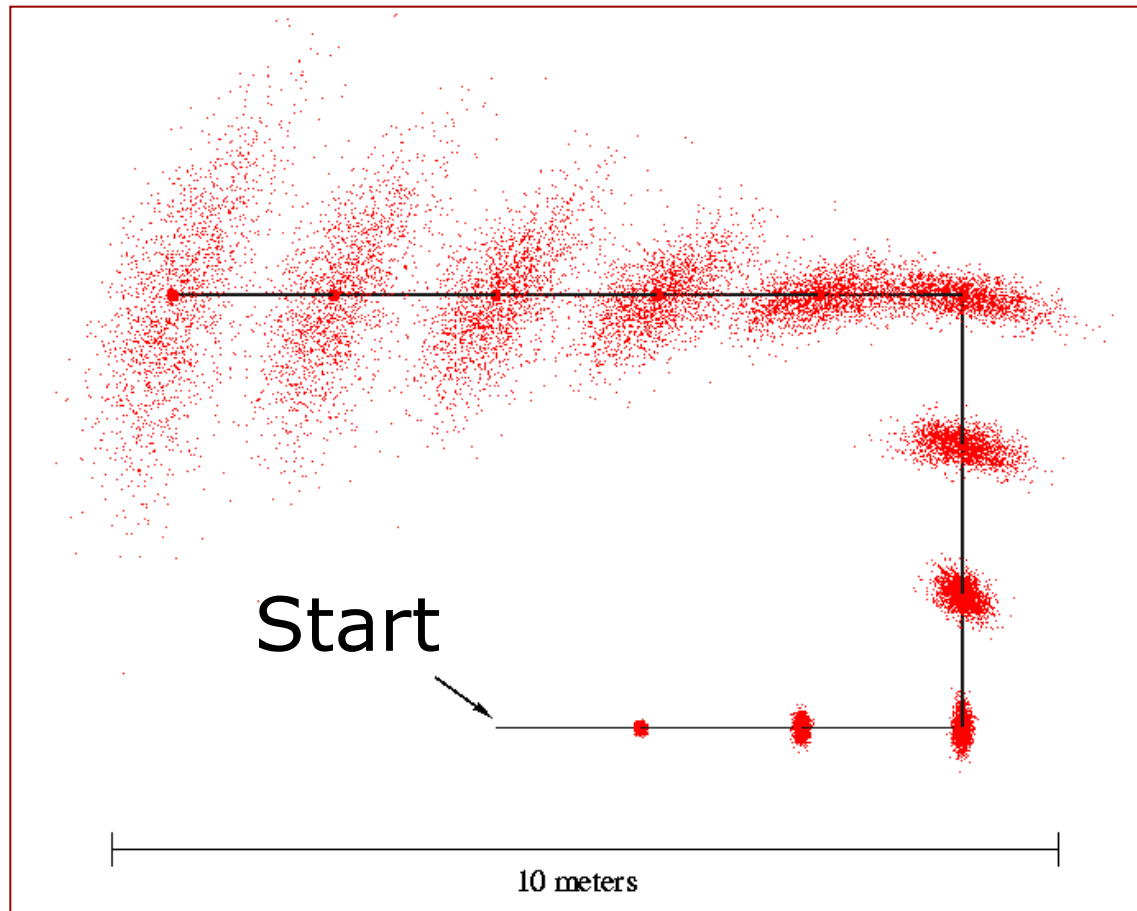
sample_normal_distribution



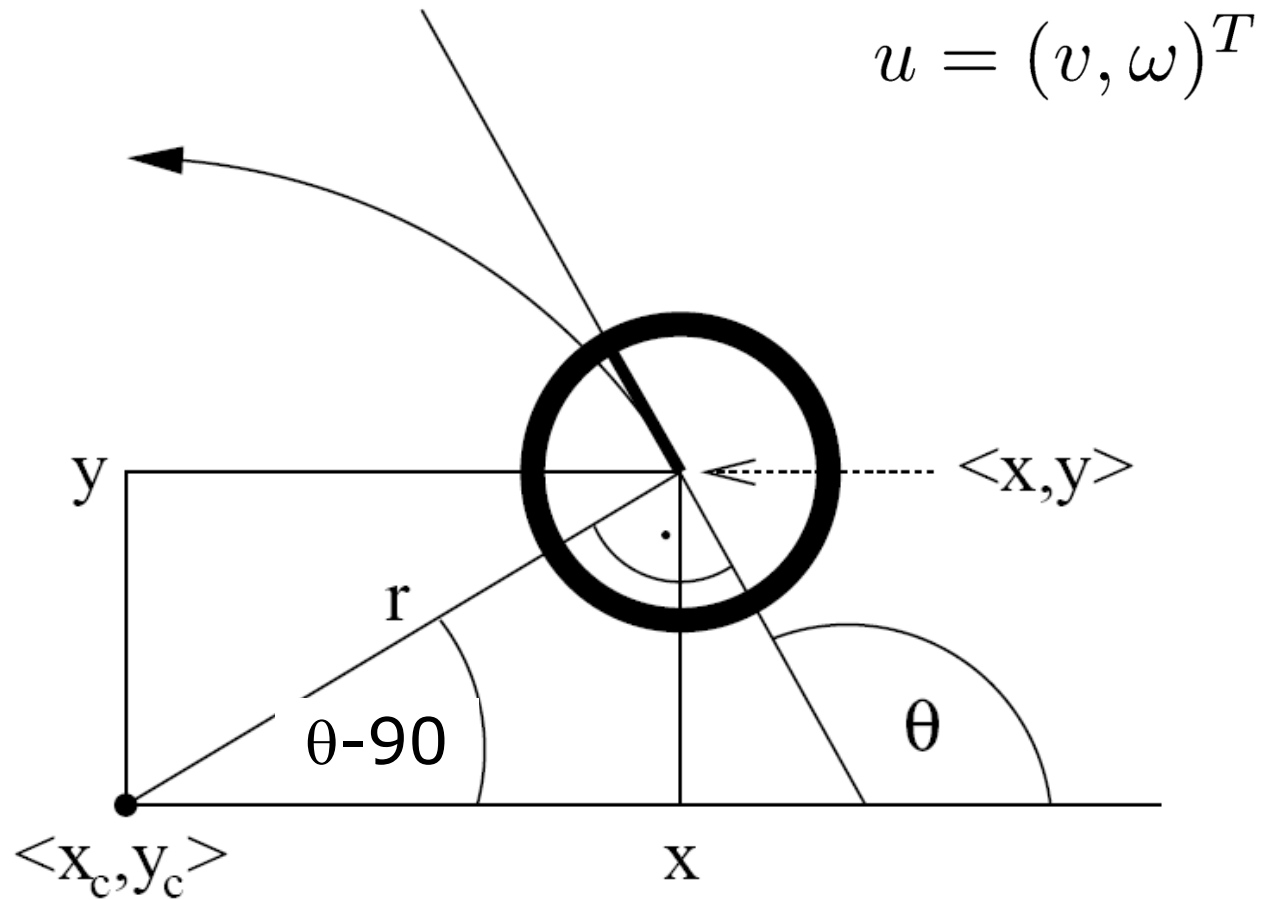
Examples (Odometry-Based)



Sampling from Our Motion Model



Velocity-Based Model



Noise Model for the Velocity-Based Model

- The measured motion is given by the true motion corrupted with noise.

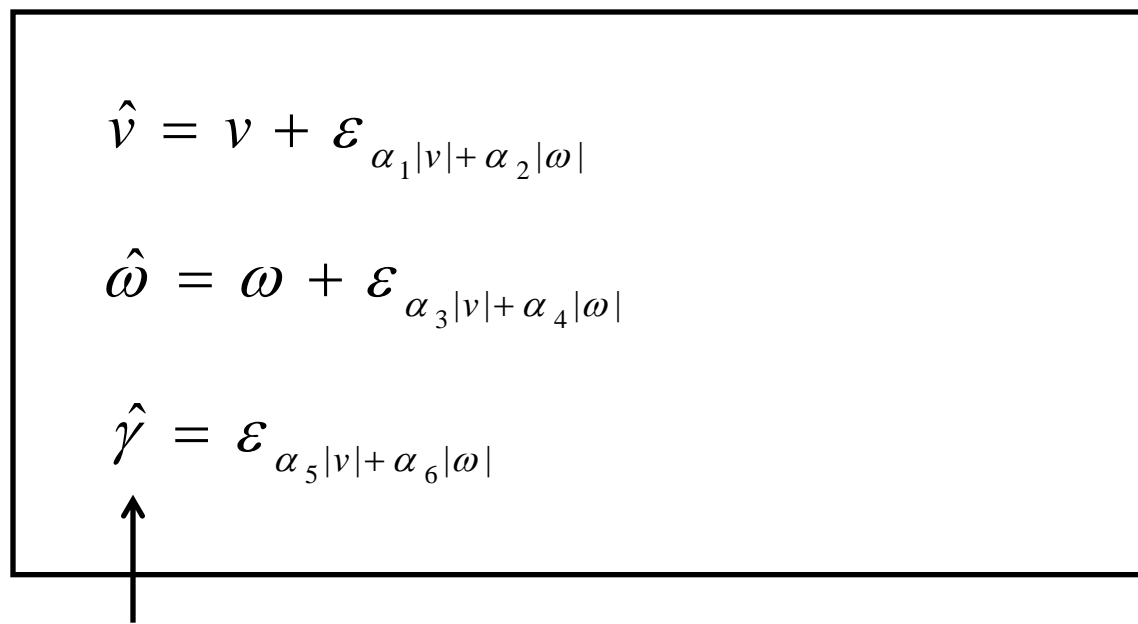
$$\hat{v} = v + \varepsilon_{\alpha_1|v| + \alpha_2|\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3|v| + \alpha_4|\omega|}$$

- Discussion: What is the disadvantage of this noise model?

Noise Model for the Velocity-Based Model

- The $(\hat{v}, \hat{\omega})$ -circle constrains the final orientation (2D manifold in a 3D space)
- Better approach:

$$\hat{v} = v + \mathcal{E}_{\alpha_1|v| + \alpha_2|\omega|}$$
$$\hat{\omega} = \omega + \mathcal{E}_{\alpha_3|v| + \alpha_4|\omega|}$$
$$\hat{\gamma} = \mathcal{E}_{\alpha_5|v| + \alpha_6|\omega|}$$


Term to account for the final rotation

Motion Including 3rd Parameter

$$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega} \Delta t)$$

$$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega} \Delta t)$$

$$\theta' = \theta + \hat{\omega} \Delta t + \hat{\gamma} \Delta t$$



Term to account for the final rotation

Equation for the Velocity Model

$$\mathbf{x}_{t-1} = (x, y, \theta)^T$$

$$\mathbf{x}_t = (x', y', \theta')^T$$

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin \theta \\ \lambda \cos \theta \end{pmatrix}$$



some constant (distance to ICC)

(center of circle is orthogonal to the initial heading)

Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$

$$x_t = (x', y', \theta')^T$$

some constant

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin \theta \\ \lambda \cos \theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

some constant (the center of the circle lies on a ray half way between x and x' and is orthogonal to the line between x and x')


Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$

$$x_t = (x', y', \theta')^T$$

some constant

Center of circle:


$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin \theta \\ \lambda \cos \theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

Allows us to solve the equations to:

$$\mu = \frac{1}{2} \frac{(x-x') \cos \theta + (y-y') \sin \theta}{(y-y') \cos \theta - (x-x') \sin \theta}$$

Equation for the Velocity Model

$$\mathbf{x}_{t-1} = (x, y, \theta)^T$$

$$\mathbf{x}_t = (x', y', \theta')^T$$

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y - y') \\ \frac{y+y'}{2} + \mu(x' - x) \end{pmatrix} \quad \mu = \frac{1}{2} \frac{(x - x') \cos \theta + (y - y') \sin \theta}{(y - y') \cos \theta - (x - x') \sin \theta}$$

and

$$r^* = \sqrt{(x' - x)^2 + (y' - y)^2}$$

$$\Delta\theta = \text{atan2}(y' - y^*, x' - x^*) - \text{atan2}(y - y^*, x - x^*)$$

Equation for the Velocity Model

- The parameters of the circle:

$$r^* = \sqrt{(x' - x)^2 + (y' - y)^2}$$

$$\Delta\theta = \text{atan2}(y' - y^*, x' - x^*) - \text{atan2}(y - y^*, x - x^*)$$

- allow for computing the velocities as

$$v = \frac{\Delta\theta}{\Delta t} r^*$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

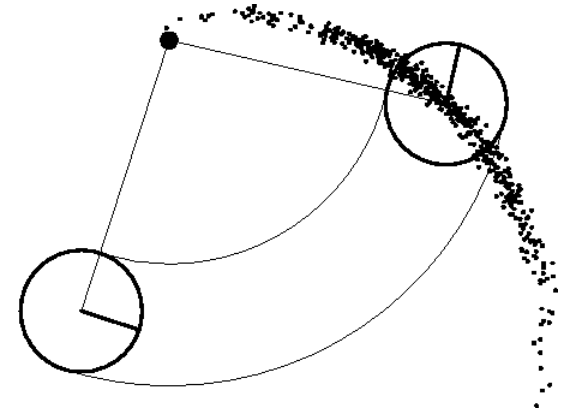
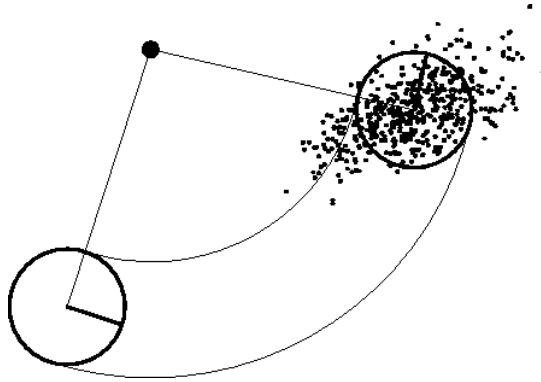
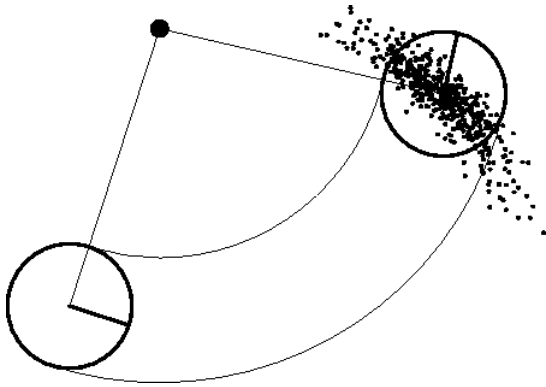
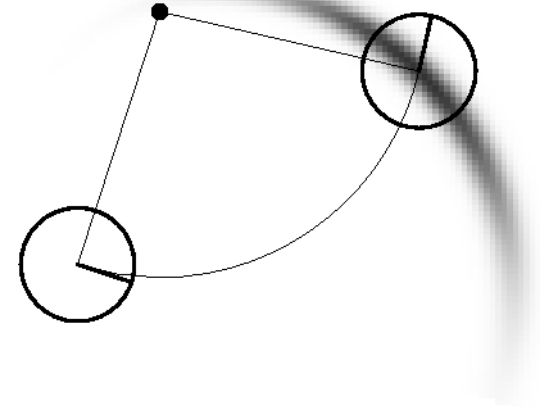
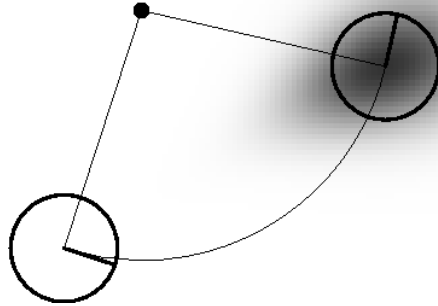
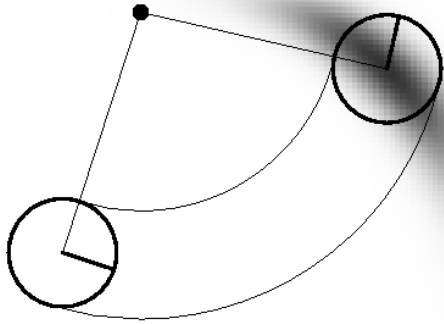
Posterior Probability for Velocity Model

- 1: **Algorithm** `motion_model_velocity`(x_t, u_t, x_{t-1}): $p(x_t \mid x_{t-1}, u_t)$
- 2:
$$\mu = \frac{1}{2} \frac{(x - x') \cos \theta + (y - y') \sin \theta}{(y - y') \cos \theta - (x - x') \sin \theta}$$
- 3:
$$x^* = \frac{x + x'}{2} + \mu(y - y')$$
- 4:
$$y^* = \frac{y + y'}{2} + \mu(x' - x)$$
- 5:
$$r^* = \sqrt{(x - x^*)^2 + (y - y^*)^2}$$
- 6:
$$\Delta\theta = \text{atan2}(y' - y^*, x' - x^*) - \text{atan2}(y - y^*, x - x^*)$$
- 7:
$$\hat{v} = \frac{\Delta\theta}{\Delta t} r^*$$
- 8:
$$\hat{\omega} = \frac{\Delta\theta}{\Delta t}$$
- 9:
$$\hat{\gamma} = \frac{\theta' - \theta}{\Delta t} - \hat{\omega}$$
- 10: **return** $\text{prob}(v - \hat{v}, \alpha_1 v^2 + \alpha_2 \omega^2) \cdot \text{prob}(\omega - \hat{\omega}, \alpha_3 v^2 + \alpha_4 \omega^2)$
 $\cdot \text{prob}(\hat{\gamma}, \alpha_5 v^2 + \alpha_6 \omega^2)$

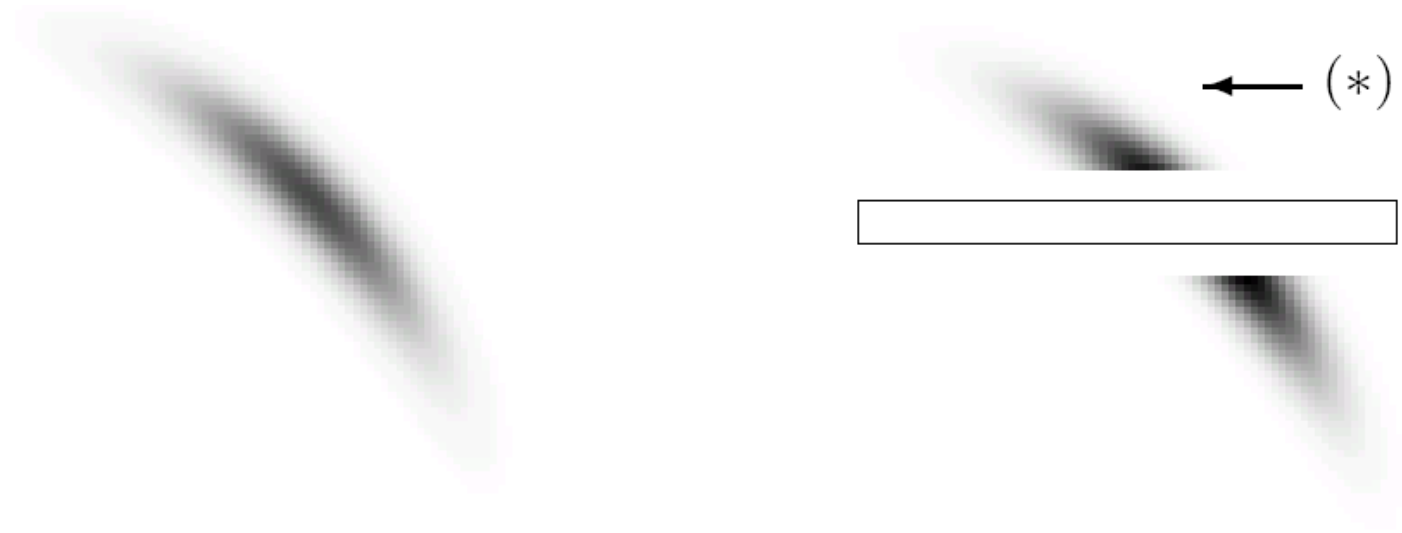
Sampling from Velocity Model

- 1: **Algorithm** `sample_motion_model_velocity`(u_t, x_{t-1}):
- 2: $\hat{v} = v + \text{sample}(\alpha_1 v^2 + \alpha_2 \omega^2)$
- 3: $\hat{\omega} = \omega + \text{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$
- 4: $\hat{\gamma} = \text{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$
- 5: $x' = x - \frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega} \Delta t)$
- 6: $y' = y + \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega} \Delta t)$
- 7: $\theta' = \theta + \hat{\omega} \Delta t + \hat{\gamma} \Delta t$
- 8: *return* $x_t = (x', y', \theta')^T$

Examples (Velocity-Based)



Map-Consistent Motion Model



$$p(x' | u, x)$$

\neq



$$p(x' | u, x, m)$$

Approximation:
$$p(x' | u, x, m) = \eta p(x' | m) p(x' | u, x)$$

Summary

- We discussed motion models for odometry-based and velocity-based systems
- We discussed ways to calculate the posterior probability $p(x' / x, u)$.
- We also described how to sample from $p(x' / x, u)$.
- Typically the calculations are done in fixed time intervals Δt .
- In practice, the parameters of the models have to be learned.
- We also discussed how to improve this motion model to take the map into account.