Introduction to Mobile Robotics

Path and Motion Planning

Wolfram Burgard, Diego Tipaldi, Barbara Frank

Motion Planning

Latombe (1991):

"…eminently necessary since, by definition, a robot accomplishes tasks by moving in the real world."

Goals:

- Collision-free trajectories.
- Robot should reach the goal location as fast as possible.

… in Dynamic Environments

- How to react to unforeseen obstacles?
  - efficiency
  - reliability

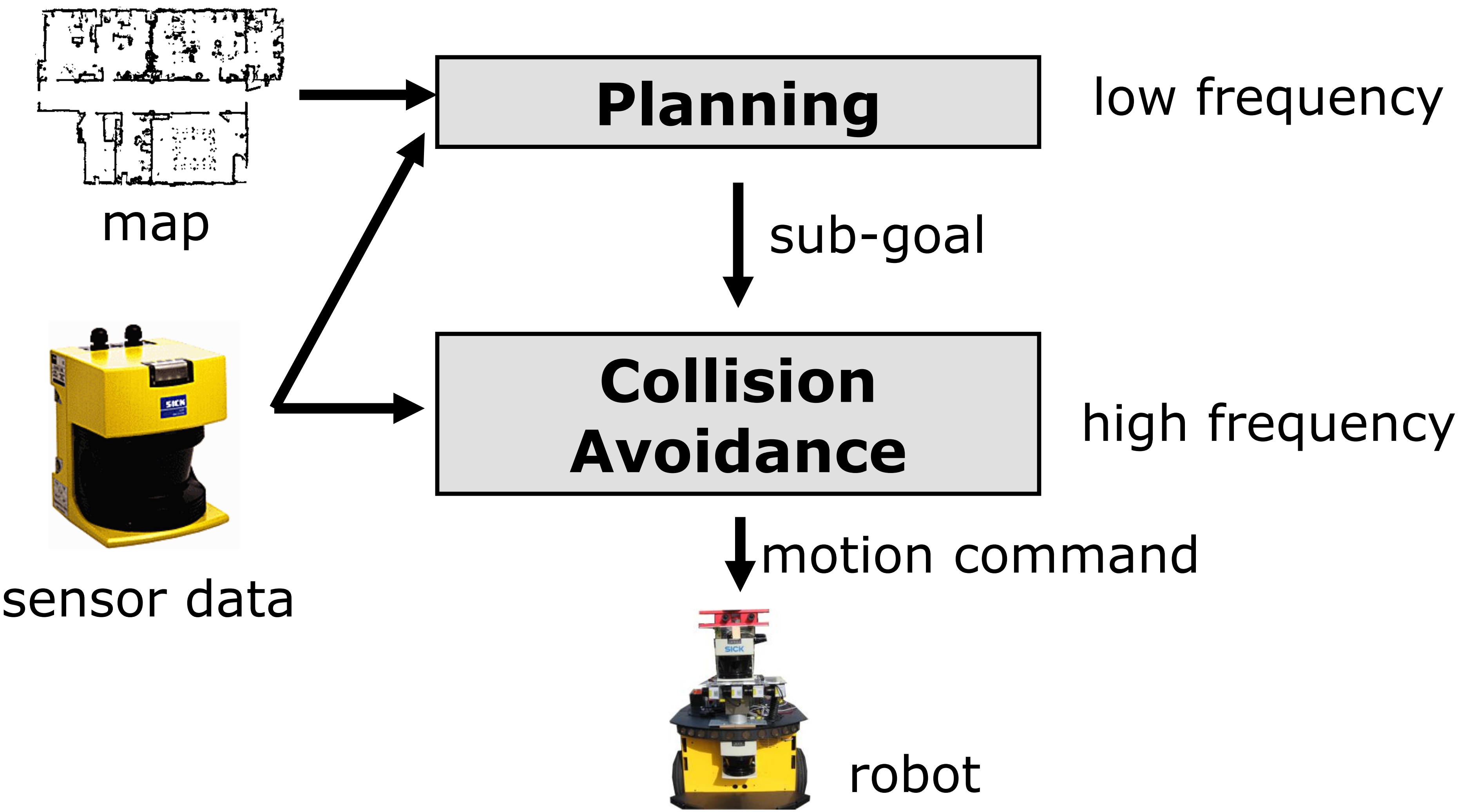
- Dynamic Window Approaches
  [Simmons, 96], [Fox et al., 97], [Brock & Khatib, 99]
- Grid map based planning
  [Konolige, 00]
- Nearness Diagram Navigation
  [Minguez at al., 2001, 2002]
- Vector-Field-Histogram+
  [Ulrich & Borenstein, 98]
- A*, D*, D* Lite, ARA*, …

Two Challenges

- Calculate the optimal path taking potential uncertainties in the actions into account

- Quickly generate actions in the case of unforeseen objects

Classic Two-layered Architecture



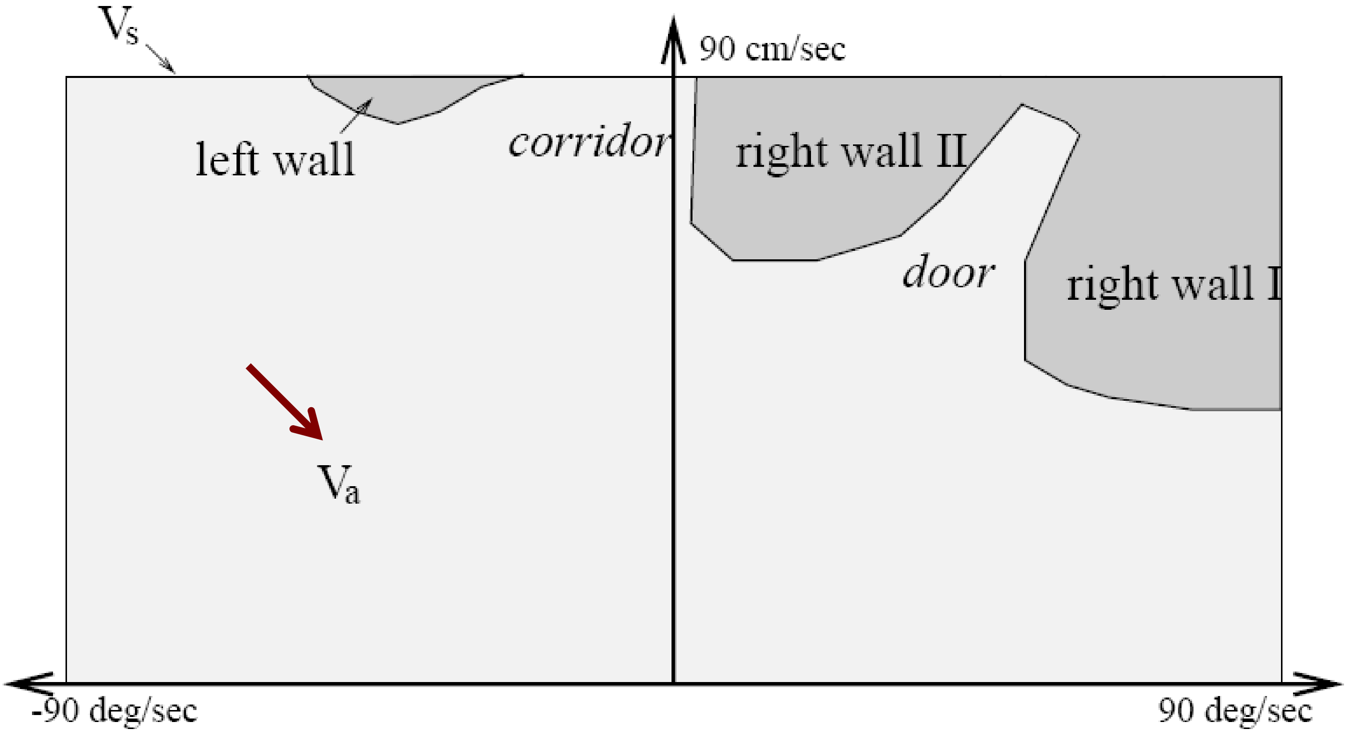
Dynamic Window Approach

- **Collision avoidance:** Determine collision-free trajectories using geometric operations
- Here: Robot moves on circular arcs
- Motion commands (v,ω)
- Which (v,ω) are admissible and reachable?

Admissible Velocities

- Speeds are admissible if the robot would be able to stop before reaching the obstacle

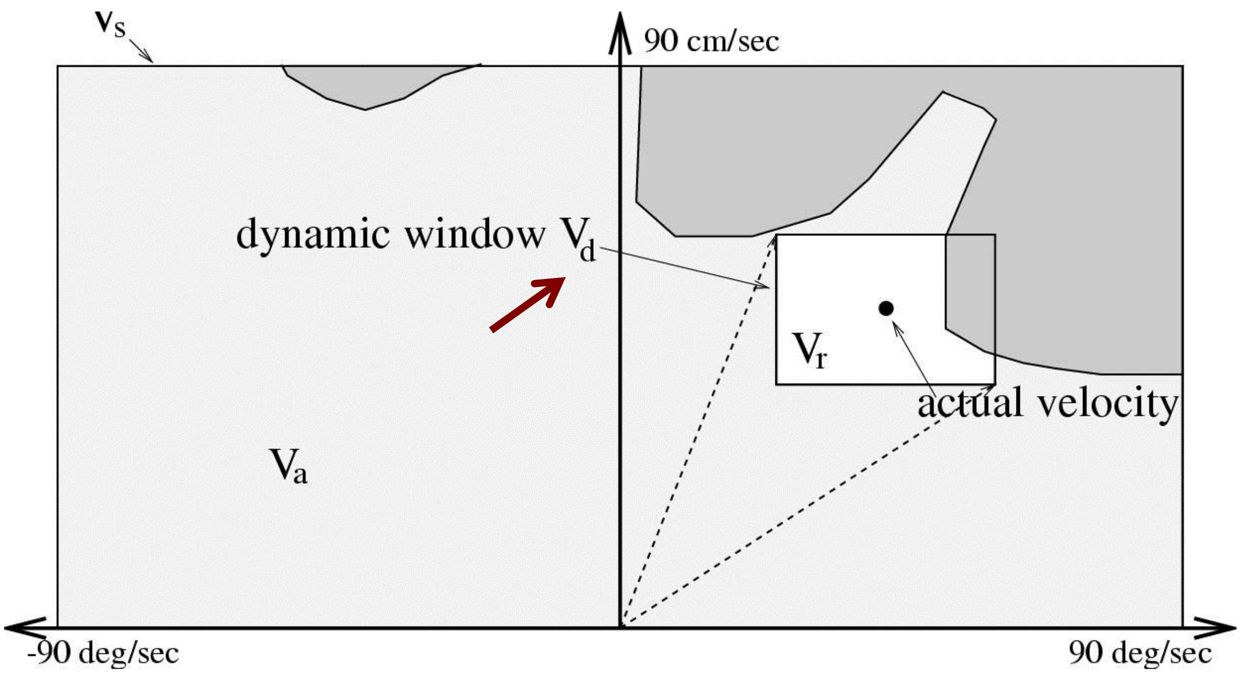
$$V_a = \{(v,\omega) \mid v \leq \sqrt{2\text{dist}(v,\omega)a_{trans}} \wedge \omega \leq \sqrt{2\text{dist}(v,\omega)a_{rot}}\}$$



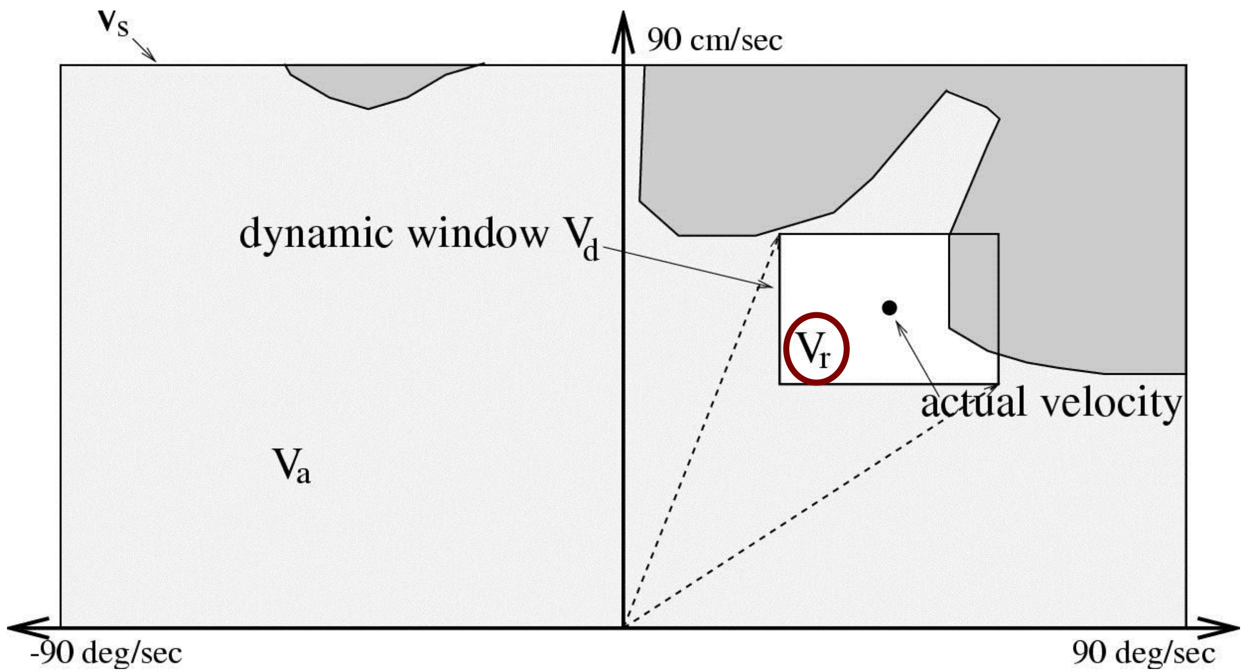
Reachable Velocities

- Speeds that are reachable by acceleration

$$V_d = \{(v, \omega) \;\; | \;\; v \in [v - a_{trans}t, v + a_{trans}t] \;\; \wedge$$
$$\omega \in [\omega - a_{rot}t, \omega + a_{rot}t]\}$$



DWA Search Space



- V_s = all possible speeds of the robot.
- V_a = obstacle free area.
- V_d = speeds reachable within a certain time frame based on possible accelerations.

$$V_r = V_s \cap V_a \cap V_d$$

Dynamic Window Approach

- How to choose <v,ω>?
- Steering commands are chosen by a heuristic navigation function.
- This function tries to minimize the travel-time by:
  "**driving fast** in the **right direction**."

Dynamic Window Approach

- Heuristic navigation function.
- Planning restricted to <x,y>-space.
- No planning in the velocity space.

Navigation Function: [Brock & Khatib, 99]

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

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Maximizes velocity.

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Maximizes velocity.

Considers cost to reach the goal.

Dynamic Window Approach

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Maximizes velocity.

Considers cost to reach the goal.

Follows grid based path computed by A*.

Dynamic Window Approach

- Heuristic navigation function.
- Planning restricted to <x,y>-space.
- No planning in the velocity space.

Navigation Function:

Goal nearness.

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Maximizes velocity.

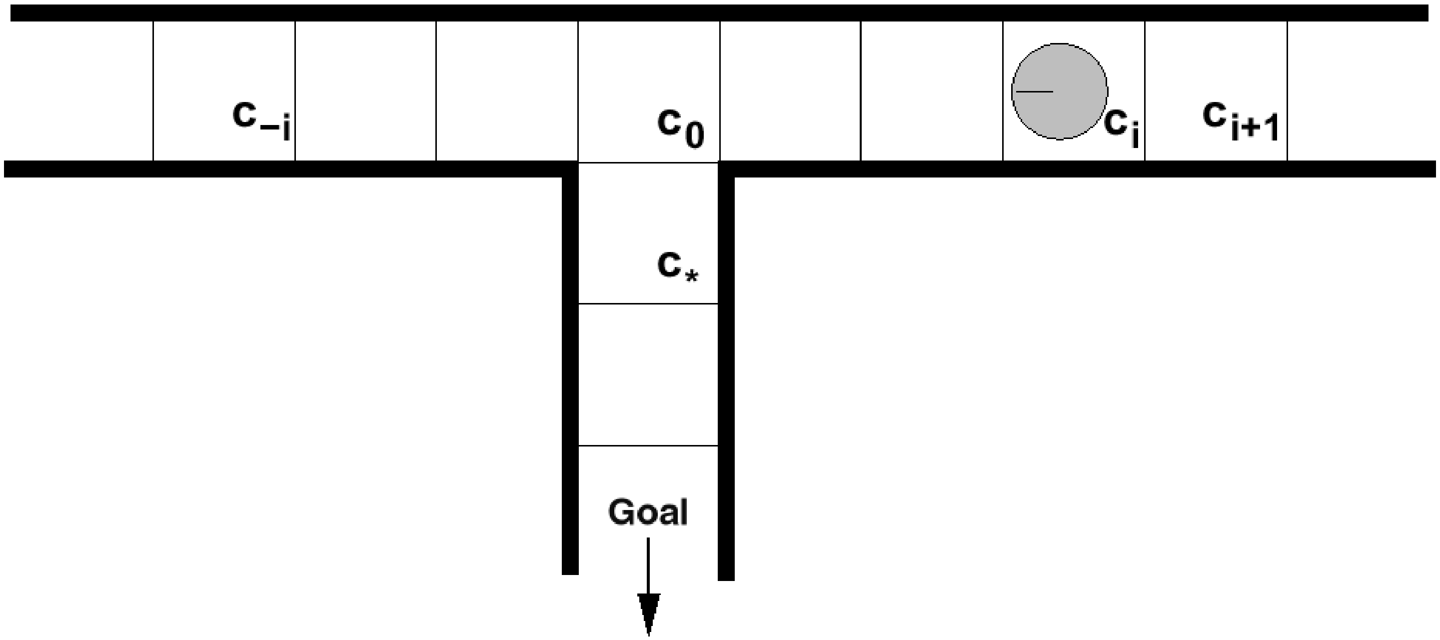
Considers cost to reach the goal.

Follows grid based path computed by A*.

Dynamic Window Approach

- Reacts quickly.
- Low CPU power requirements.
- Guides a robot on a collision-free path.
- Successfully used in a lot of real-world scenarios.
- Resulting trajectories sometimes sub-optimal.
- Local minima might prevent the robot from reaching the goal location.

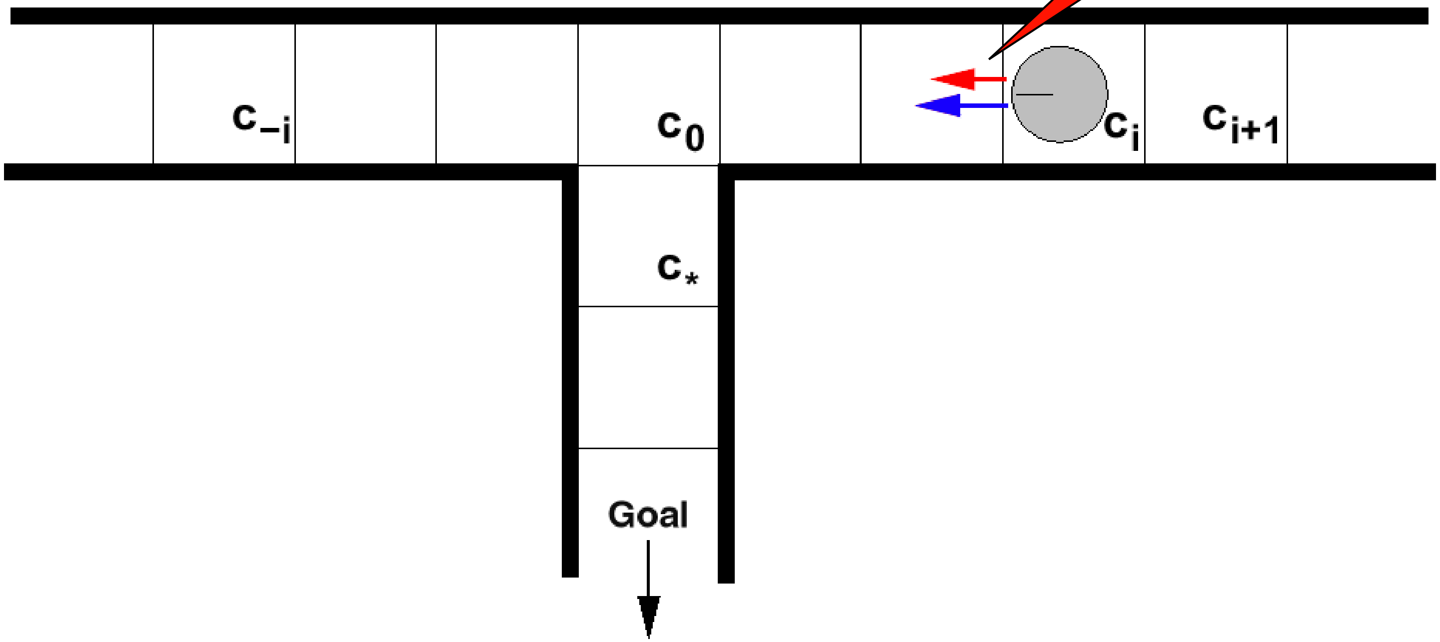
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs

Robot 's velocity.



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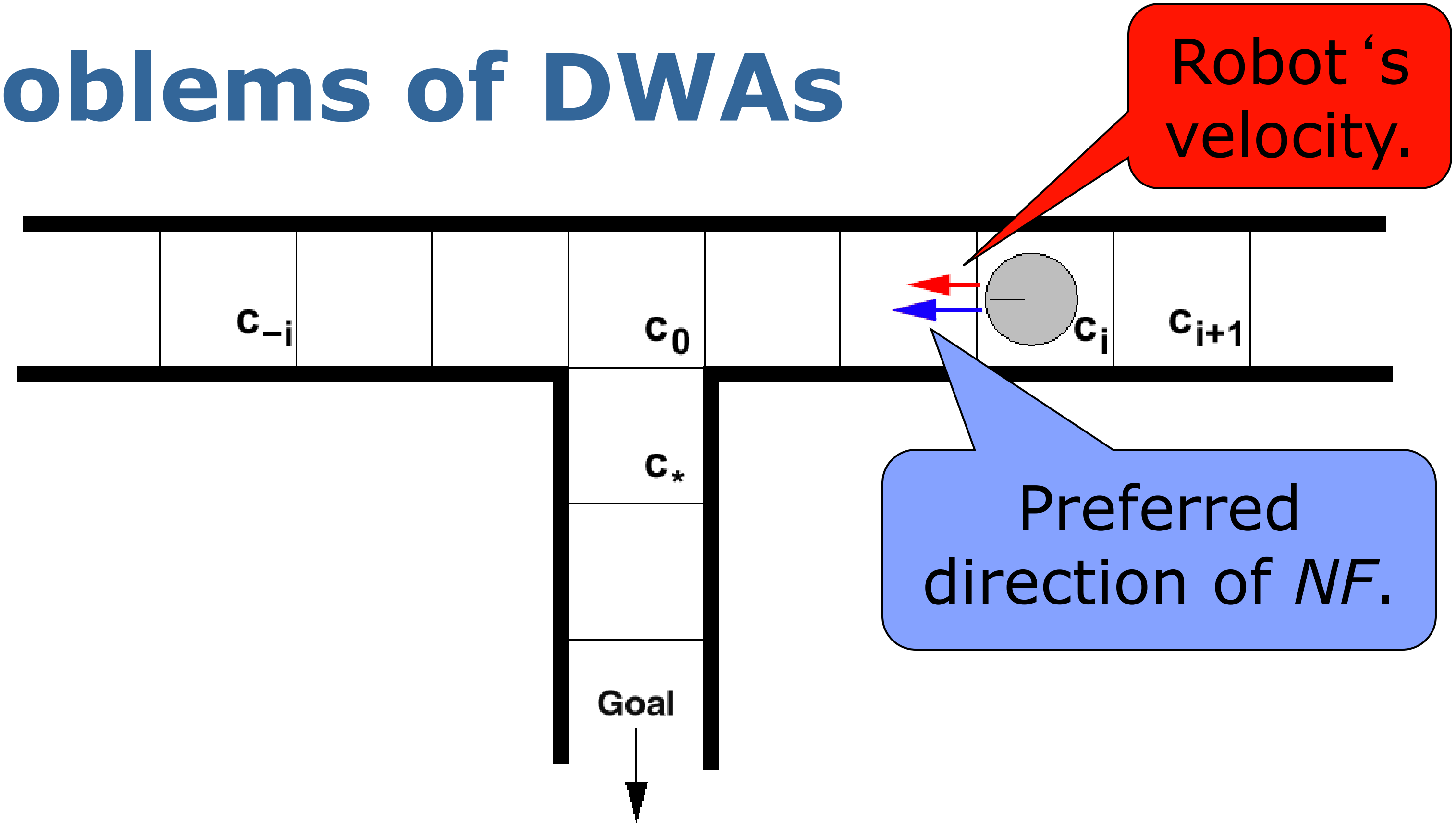
Problems of DWAs



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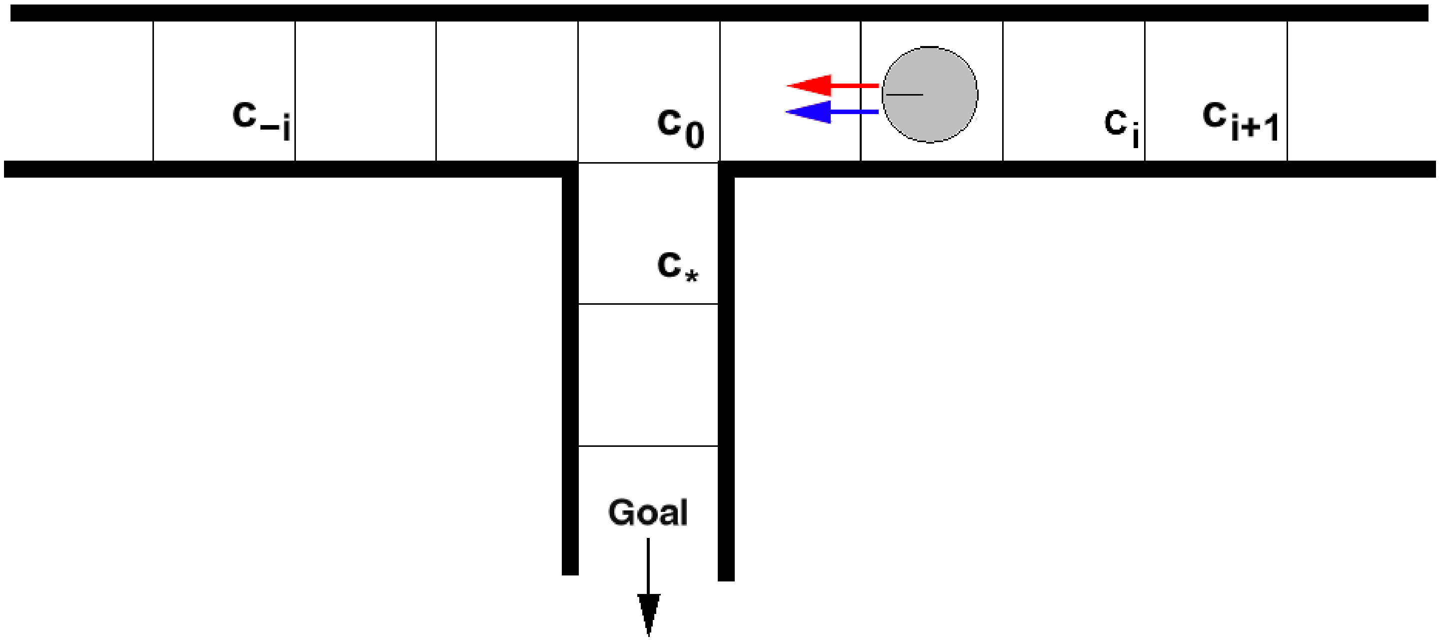

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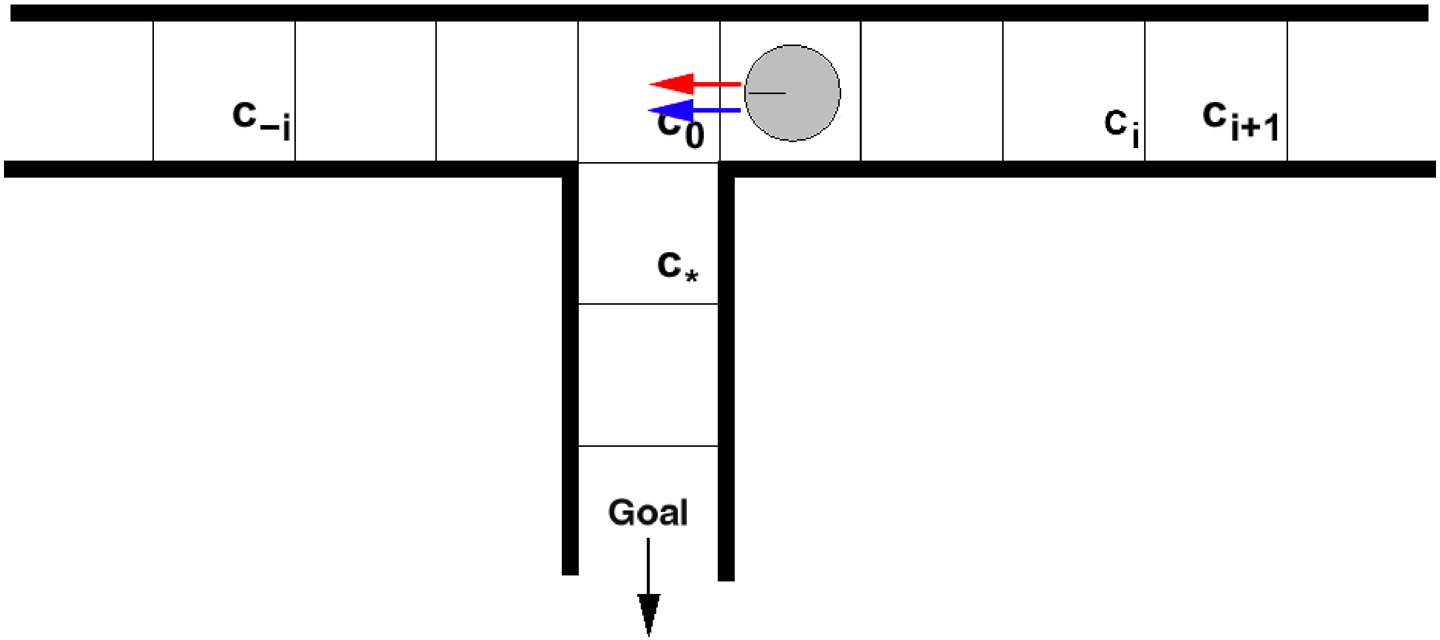

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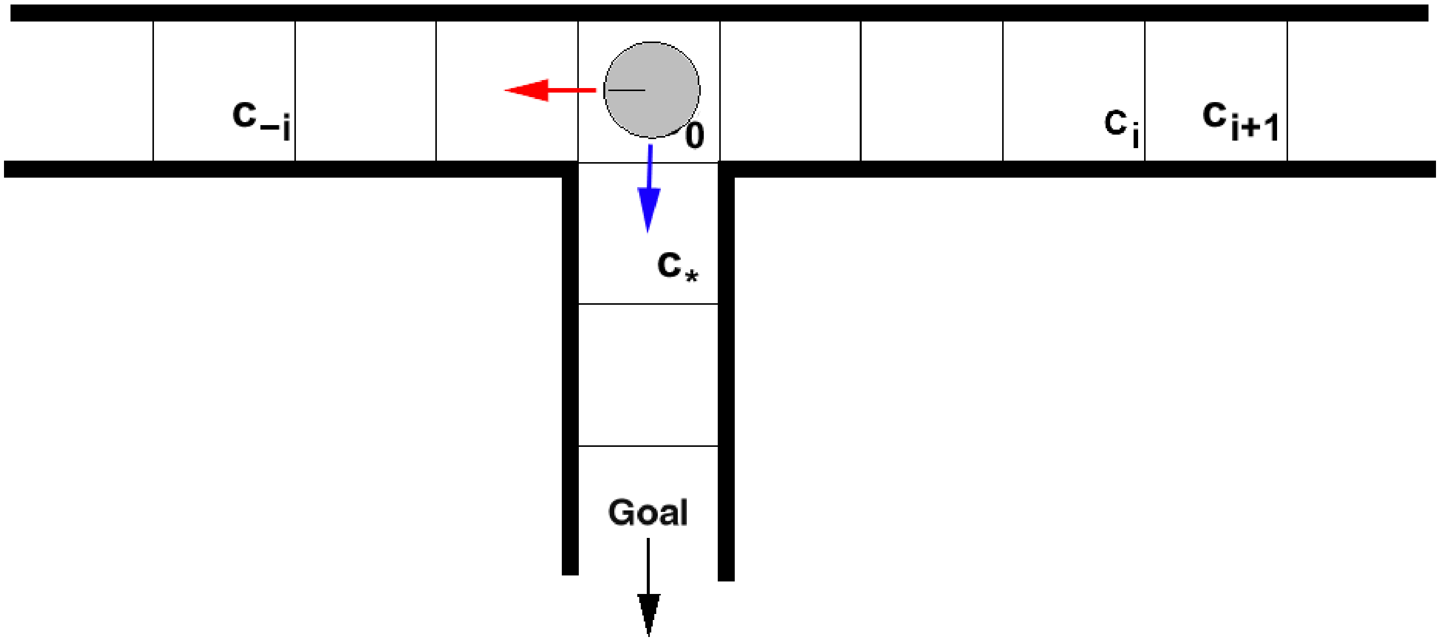
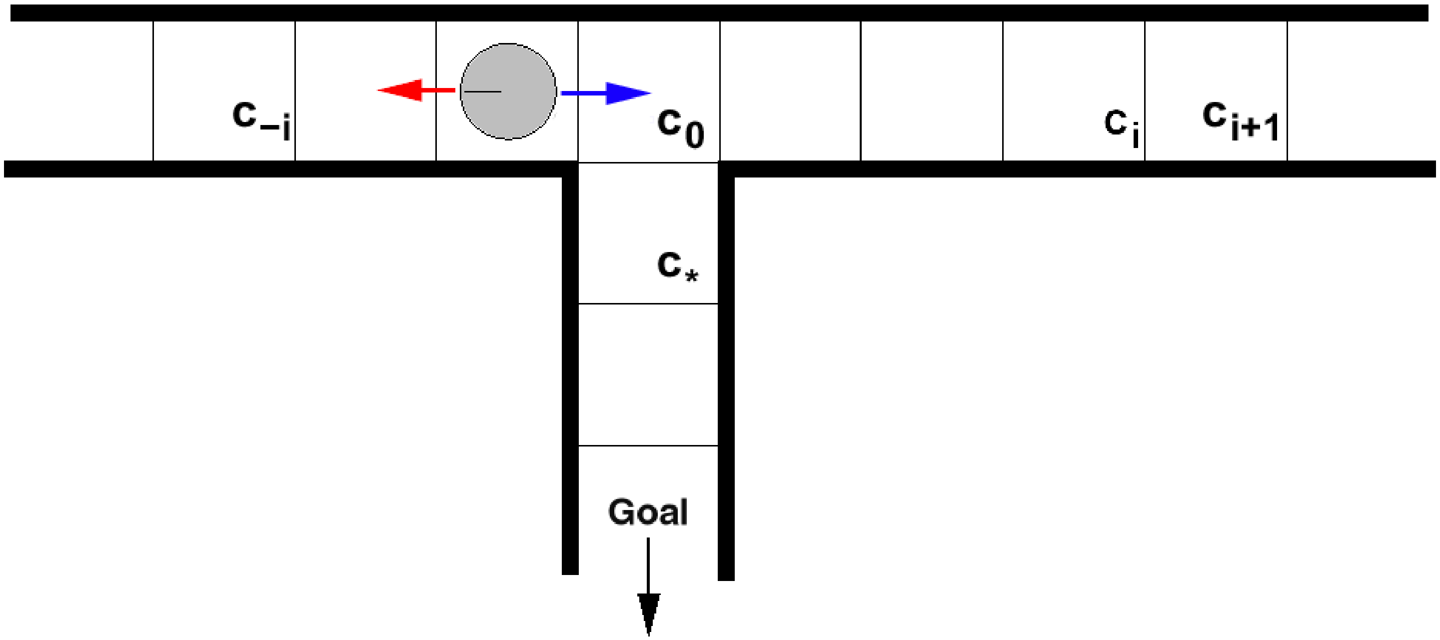
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

- The robot drives too fast at c_0 to enter corridor facing south.

Problems of DWAs



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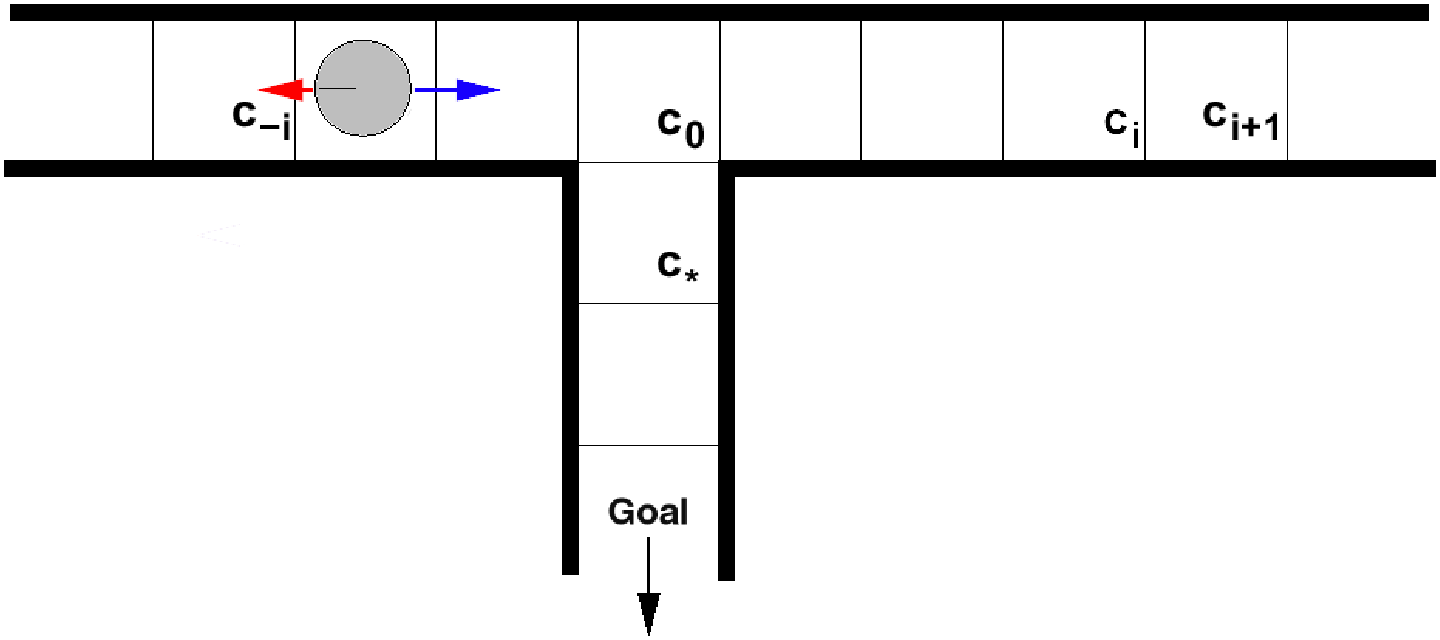
Problems of DWAs



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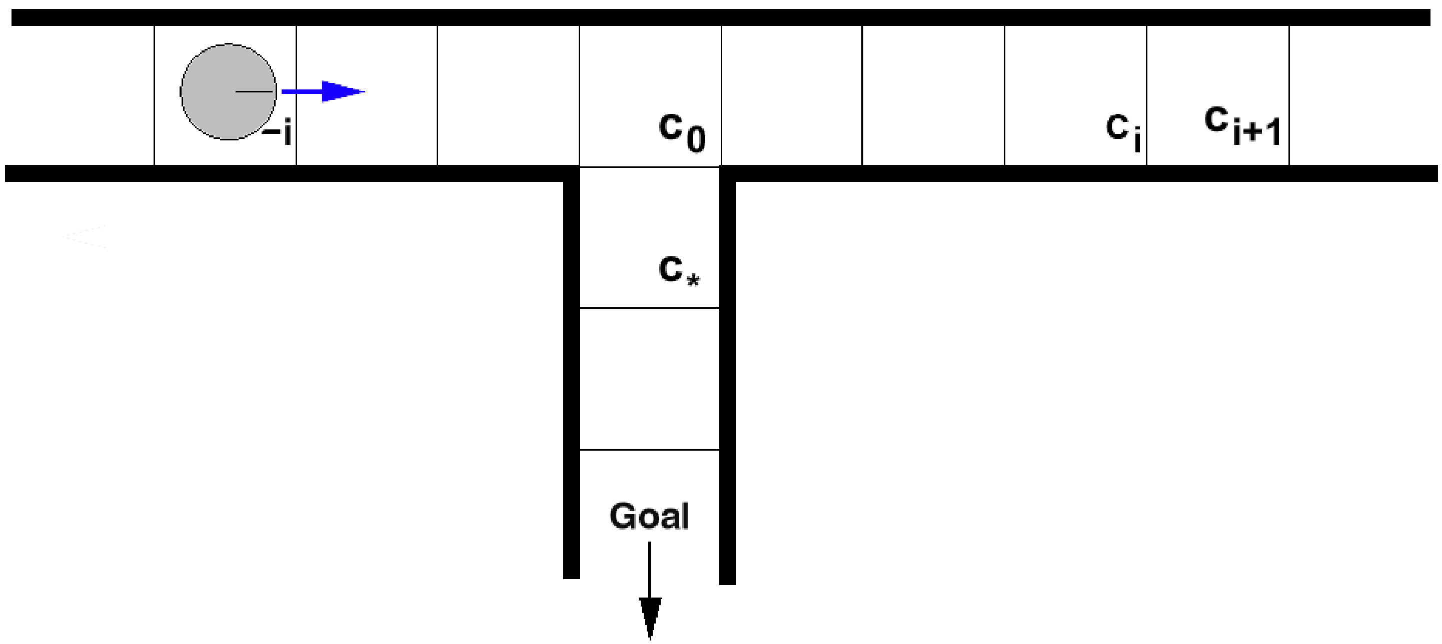
Problems of DWAs



- Same situation as in the beginning.

  ➔ DWAs have problems to reach the goal.

Problems of DWAs

- Typical problem in a real world situation:

- Robot does not slow down early enough to enter the doorway.

Motion Planning Formulation

- The **problem of motion planning** can be stated as follows. Given:
  - A **start** pose of the robot
  - A desired **goal** pose
  - A geometric description of the **robot**
  - A geometric description of the **world**

- Find a path that moves the robot gradually from **start** to **goal** while **never touching** any obstacle

Configuration Space

- Although the motion planning problem is defined in the regular world, it lives in another space: the **configuration space**

- A robot configuration q is a specification of the positions of all robot points relative to a fixed coordinate system

- Usually a configuration is expressed as a **vector of positions** and **orientations**

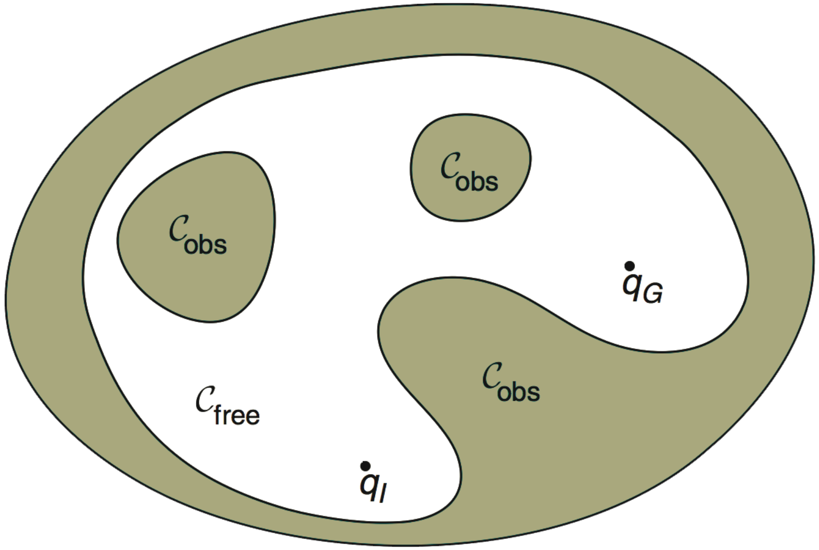
Configuration Space

Free space and **obstacle region**

- With $\mathcal{W} = \mathbb{R}^m$ being the work space, $\mathcal{O} \in \mathcal{W}$ the set of obstacles, $\mathcal{A}(q)$ the robot in configuration $q \in \mathcal{C}$

$$\begin{aligned} \mathcal{C}_{free} &= \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} = \emptyset\} \\ \mathcal{C}_{obs} &= \mathcal{C}/\mathcal{C}_{free} \end{aligned}$$

- We further define
  q_I : start configuration
  q_G: goal configuration



Configuration Space

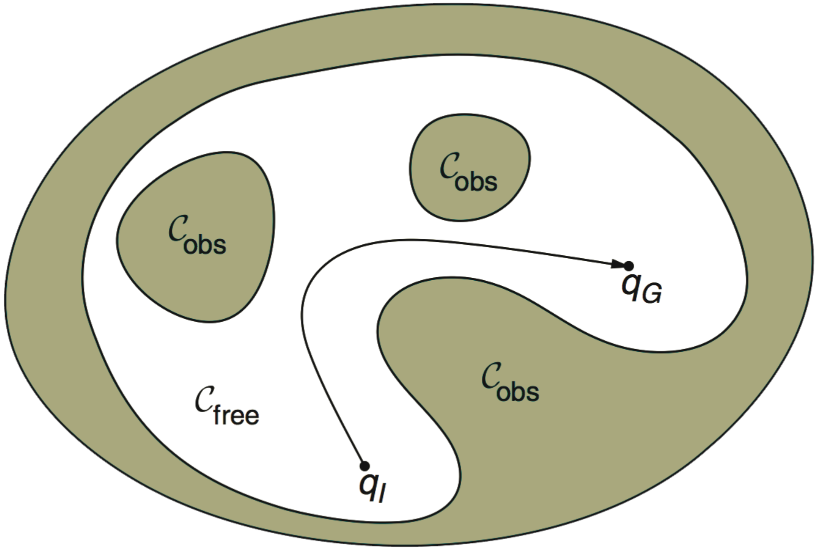
Then, motion planning amounts to

- Finding a continuous path

  $$\tau : [0, 1] \rightarrow \mathcal{C}_{free}$$

  with $\tau(0) = q_I \, , \, \tau(1) = q_G$

- Given this setting, we can do planning with the robot being a **point in C-space!**



C-Space Discretizations

- Continuous terrain needs to be **discretized** for path planning
- There are **two general approaches** to discretize C-spaces:
  - **Combinatorial planning**
    Characterizes C_{free} explicitely by capturing the connectivity of C_{free} into a graph and finds solutions using search
  - **Sampling-based planning**
    Uses collision-detection to probe and incrementally search the C-space for solution

Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Uninformed search:** besides the problem definition, no further information about the domain („blind search")
- The only thing one can do is to expand nodes differently
- Example algorithms: breadth-first, uniform-cost, depth-first, bidirectional, etc.

Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

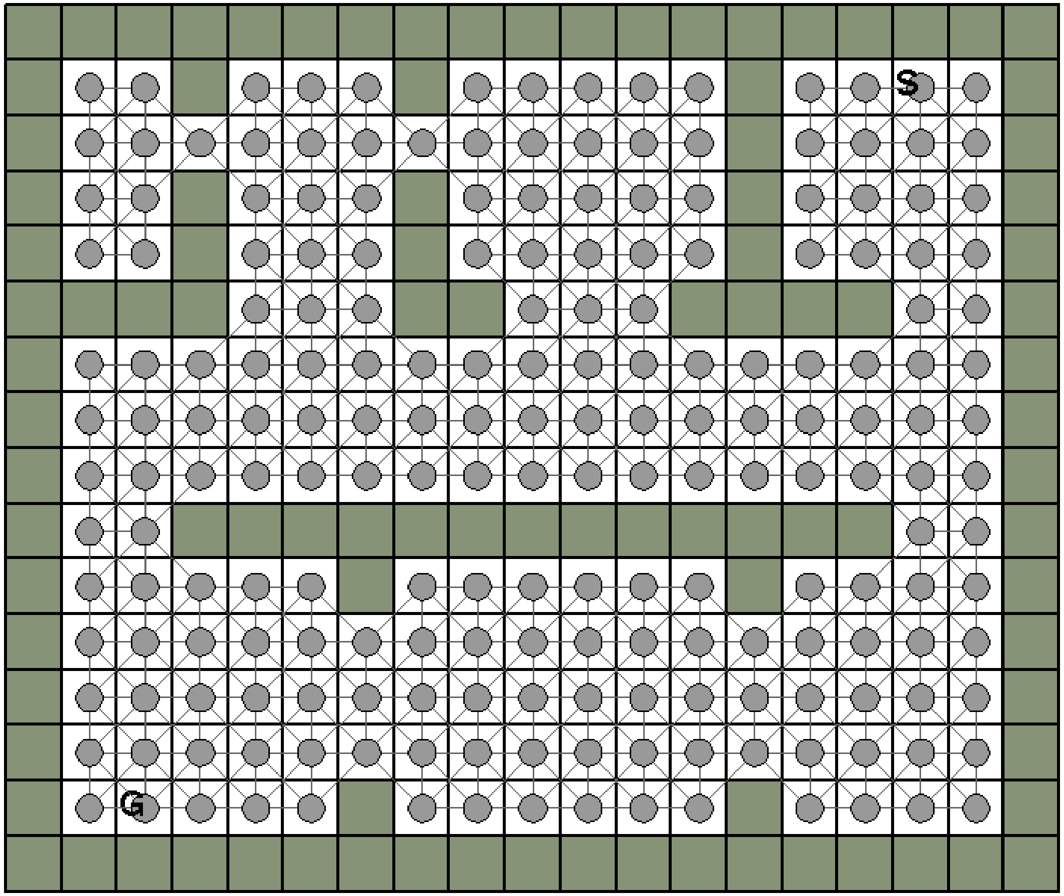
- **Informed search:** further information about the domain through **heuristics**
- Capability to say that a node is "more promising" than another node
- Example algorithms: greedy best-first search, **A***, many variants of A*, D*, etc.

Search

The performance of a search algorithm is measured in four ways:

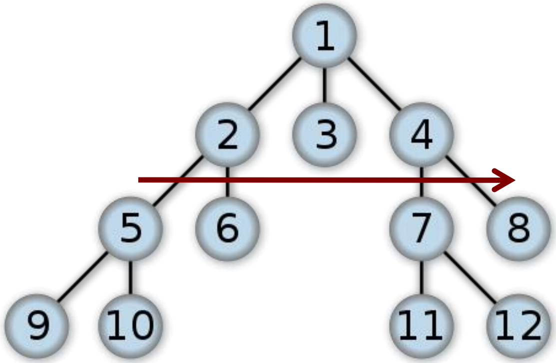
- **Completeness:** does the algorithm find the solution when there is one?
- **Optimality:** is the solution the best one of all possible solutions in terms of path cost?
- **Time complexity:** how long does it take to find a solution?
- **Space complexity:** how much memory is needed to perform the search?

Discretized Configuration Space

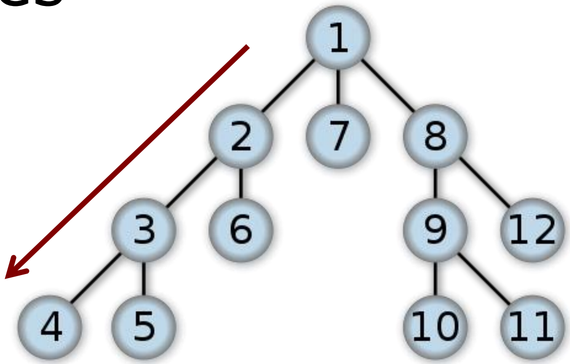


Uninformed Search

- **Breadth-first**
  - Complete
  - Optimal if action costs equal
  - Time and space: $O(b^d)$



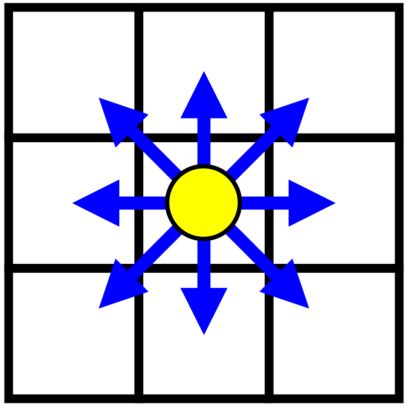
- **Depth-first**
  - Not complete in infinite spaces
  - Not optimal
  - Time: $O(b^m)$
  - Space: $O(bm)$ (can forget explored subtrees)



(b: branching factor, d: goal depth, m: max. tree depth)

Informed Search: A*

- What about using A* to plan the path of a robot?
- Finds the shortest path
- Requires a graph structure
- Limited number of edges
- In robotics: planning on a 2d occupancy grid map



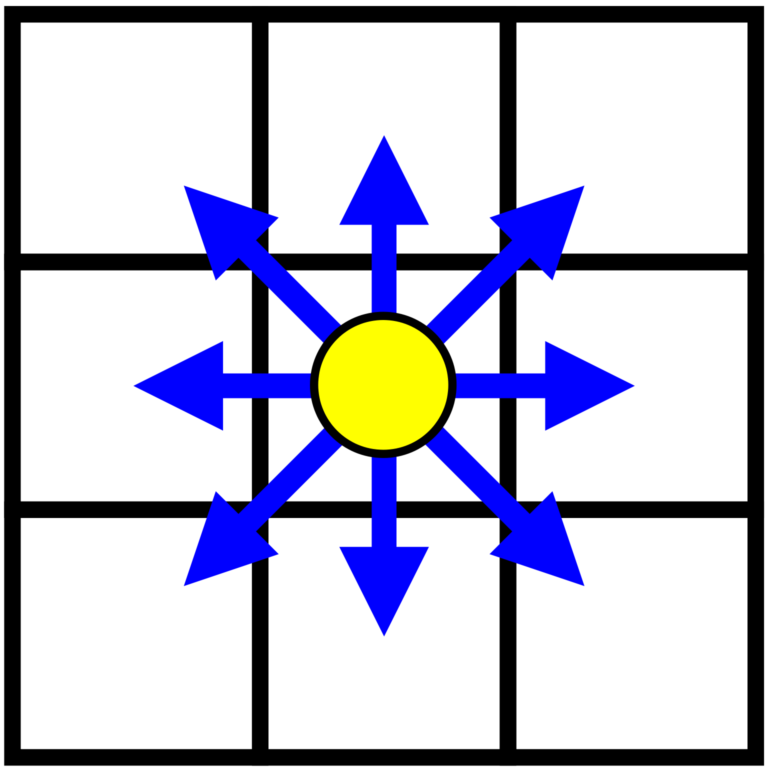
A*: Minimize the estimated path costs

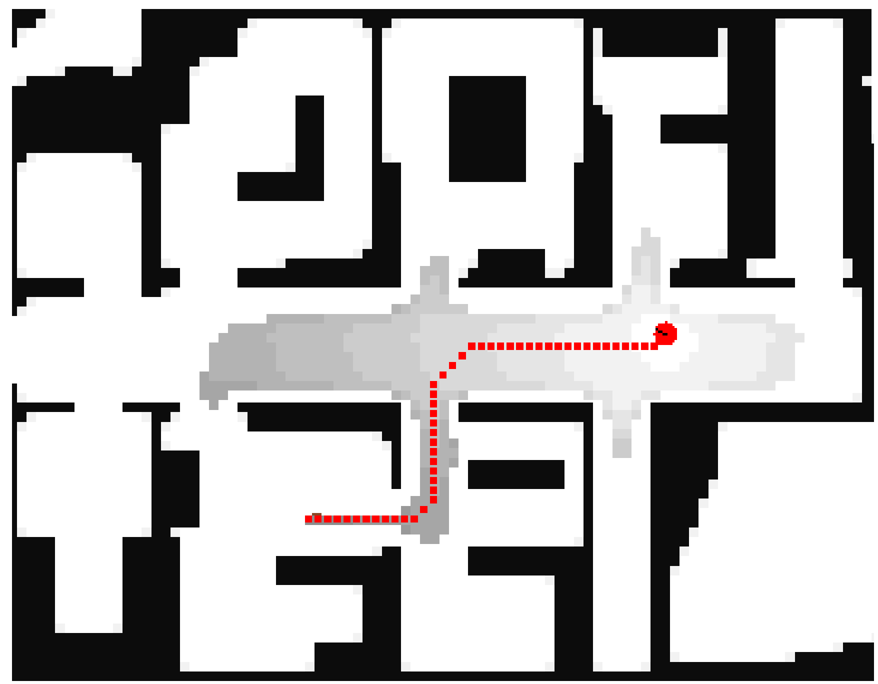
- $g(n)$ = actual cost from the initial state to n.
- $h(n)$ = estimated cost from n to the next goal.
- $f(n) = g(n) + h(n)$, the estimated cost of the cheapest solution through n.
- Let h*(n) be the actual cost of the optimal path from n to the next goal.
- h is admissible if the following holds for all n :

$$h(n) \leq h^*(n)$$

- We require that for A*, h is admissible (the straight-line distance is admissible in the Euclidean Space).

Example: Path Planning for Robots in a Grid-World





Deterministic Value Iteration

- To compute the shortest path from every state to one goal state, use (deterministic) value iteration.
- Very similar to Dijkstra's Algorithm.
- Such a cost distribution is the optimal heuristic for A*.

Typical Assumption in Robotics for A* Path Planning

- The robot is assumed to be localized.
- The robot computes its path based on an occupancy grid.
- The correct motion commands are executed.

Is this always true?

Problems

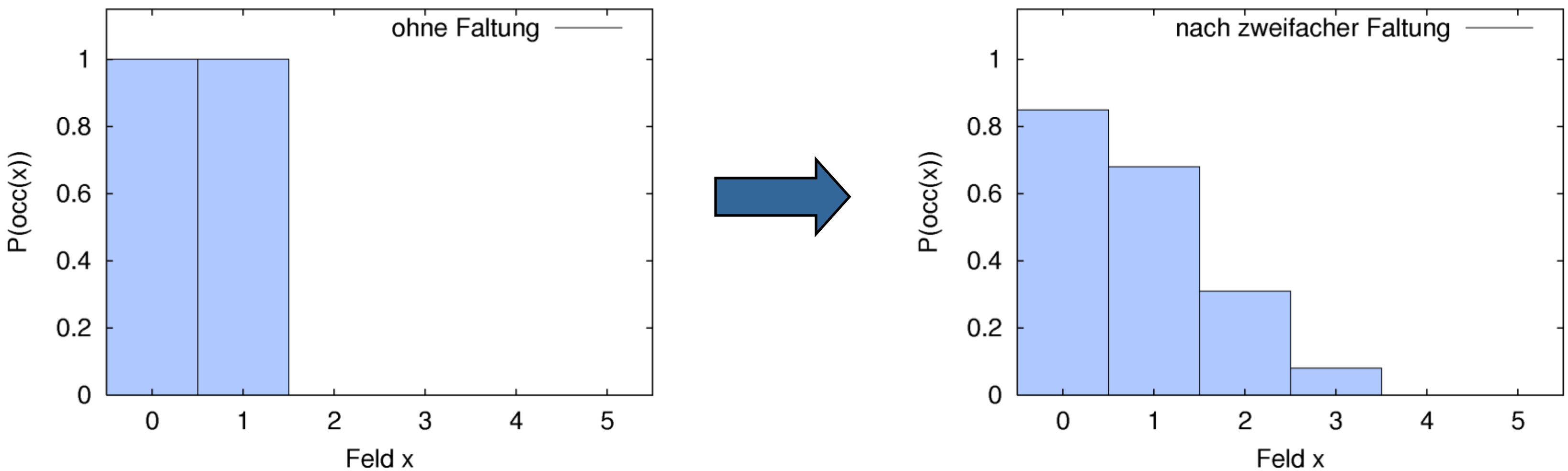
- What if the robot is slightly delocalized?
- Moving on the shortest path guides often the robot on a trajectory close to obstacles.
- Trajectory aligned to the grid structure.

Convolution of the Grid Map

- Convolution blurs the map.
- Obstacles are assumed to be bigger than in reality.
- Perform an A* search in such a convolved map.
- Robot increases distance to obstacles and moves on a short path!

Example: Map Convolution

- 1-d environment, cells c_0, ..., c_5



- Cells before and after 2 convolution runs.

Convolution

- Consider an occupancy map. Than the convolution is defined as:

$$P(occ_{x_i,y}) = \frac{1}{4} \cdot P(occ_{x_{i-1},y}) + \frac{1}{2} \cdot P(occ_{x_i,y}) + \frac{1}{4} \cdot P(occ_{x_{i+1},y})$$

$$P(occ_{x_0,y}) = \frac{2}{3} \cdot P(occ_{x_0,y}) + \frac{1}{3} \cdot P(occ_{x_1,y})$$

$$P(occ_{x_{n-1},y}) = \frac{1}{3} \cdot P(occ_{x_{n-2},y}) + \frac{2}{3} \cdot P(occ_{x_{n-1},y})$$

- This is done for each row and each column of the map.
- "Gaussian blur"

A* in Convolved Maps

- The costs are a product of path length and occupancy probability of the cells.
- Cells with higher probability (e.g., caused by convolution) are avoided by the robot.
- Thus, it keeps distance to obstacles.
- This technique is fast and quite reliable.

5D-Planning – an Alternative to the Two-layered Architecture

- Plans in the full $\langle x,y,\theta,v,\omega \rangle$-configuration space using A^*.
  - ➔ Considers the robot's kinematic constraints.

- Generates a sequence of steering commands to reach the goal location.

- Maximizes trade-off between driving time and distance to obstacles.

The Search Space (1)

- What is a state in this space?
  $<x,y,\theta,v,\omega>$ = current position and speed of the robot

- How does a state transition look like?
  $<x_1,y_1,\theta_1,v_1,\omega_1> \rightarrow <x_2,y_2,\theta_2,v_2,\omega_2>$
  with motion command (v_2,ω_2) and
  $|v_1-v_2| < a_v$, $|\omega_1-\omega_2| < a_\omega$. Pose of the Robot is a result of the motion equations.

The Search Space (2)

Idea: search in the discretized $<x,y,\theta,v,\omega>$-space.

Problem: the search space is too huge to be explored within the time constraints (5+ Hz for online motion plannig).

Solution: restrict the full search space.

The Main Steps of the Algorithm

1. Update (static) grid map based on sensory input.

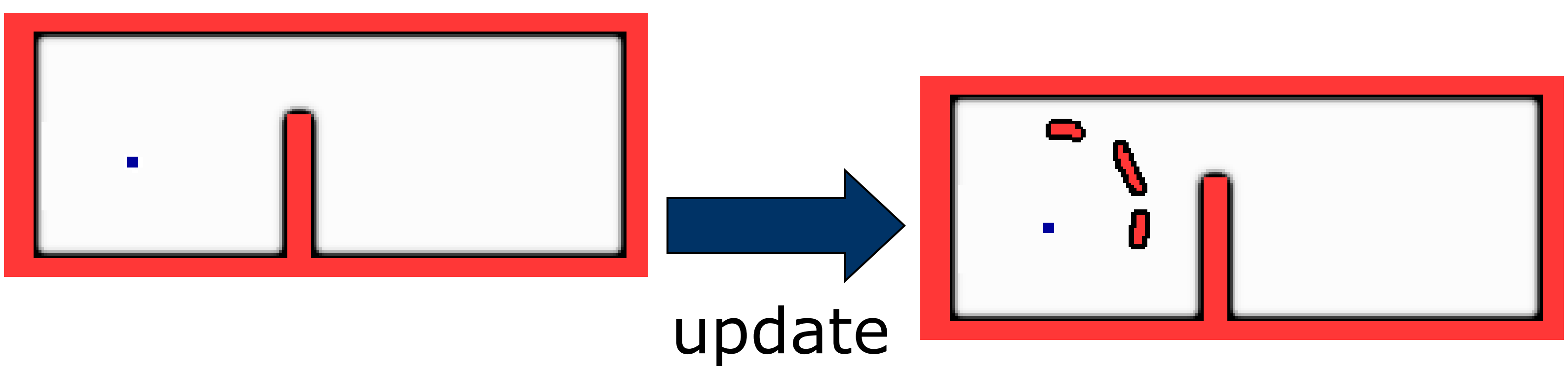
2. Use A^* to find a trajectory in the $<x,y>$-space using the updated grid map.

3. Determine a restricted 5d-configuration space based on step 2.

4. Find a trajectory by planning in the restricted $<x,y,\theta,v,\omega>$-space.

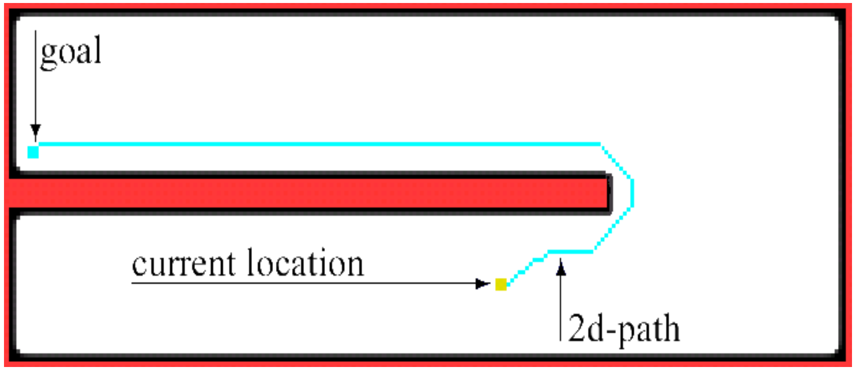
Updating the Grid Map

- The environment is represented as a 2d-occupency grid map.
- Convolution of the map increases security distance.
- Detected obstacles are added.
- Cells discovered free are cleared.



Find a Path in the 2d-Space

- Use A^* to search for the optimal path in the 2d-grid map.
- Use heuristic based on a deterministic value iteration within the static map.



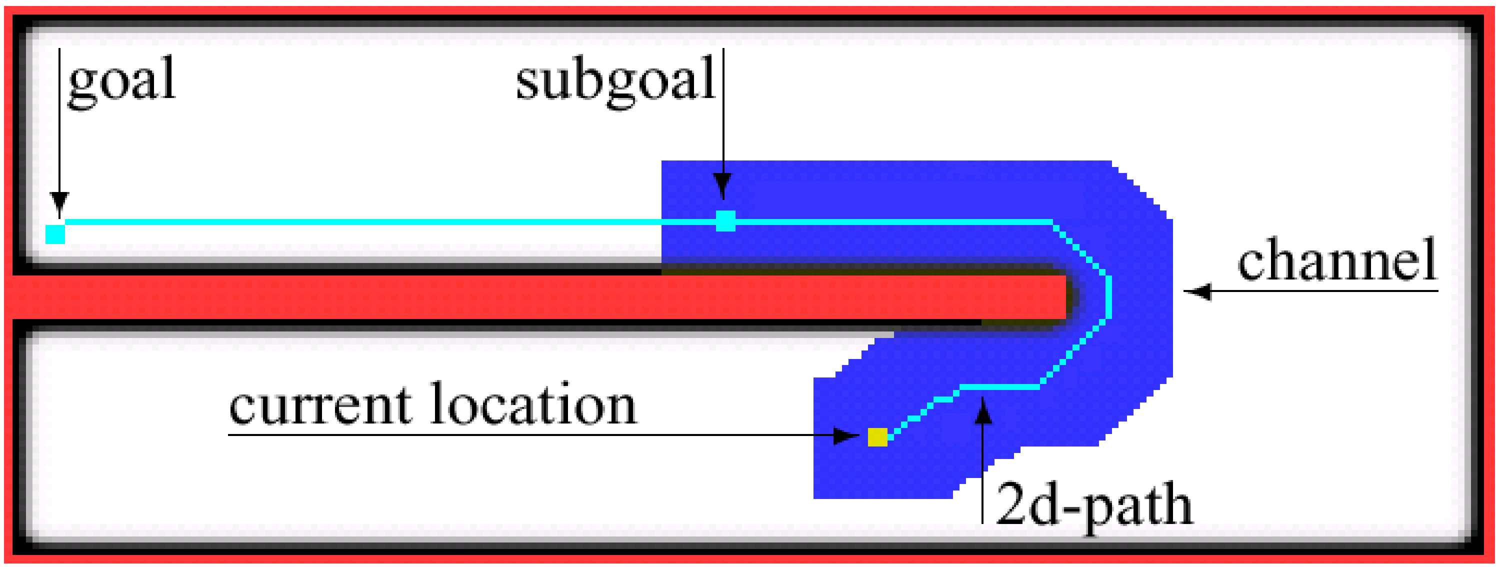
Restricting the Search Space

Assumption: the projection of the 5d-path onto the <x,y>-space lies close to the optimal 2d-path.

Therefore: construct a restricted search space (channel) based on the 2d-path.

Space Restriction

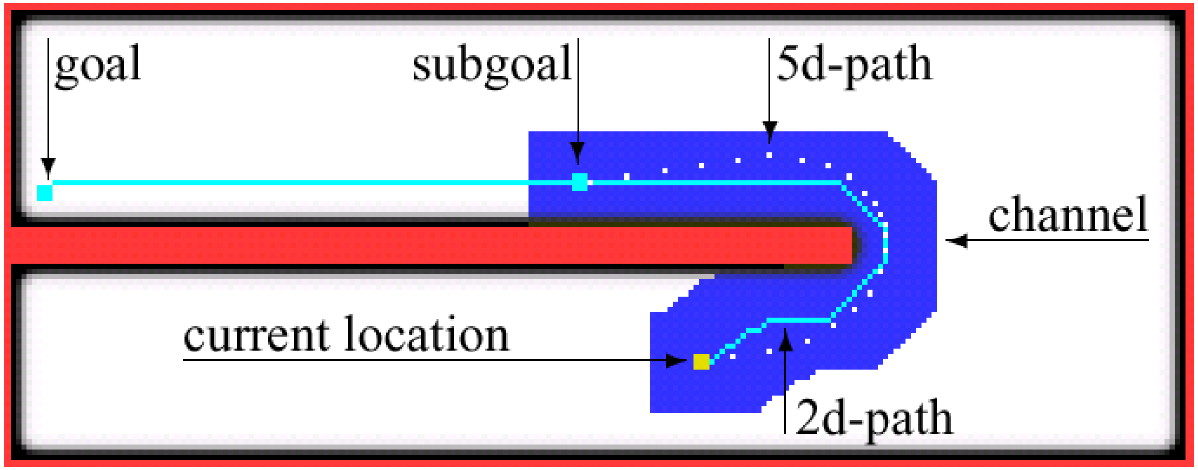
- Resulting search space = $<x, y, \theta, v, \omega>$ with $(x,y) \in$ channel.
- Choose a sub-goal lying on the 2d-path within the channel.

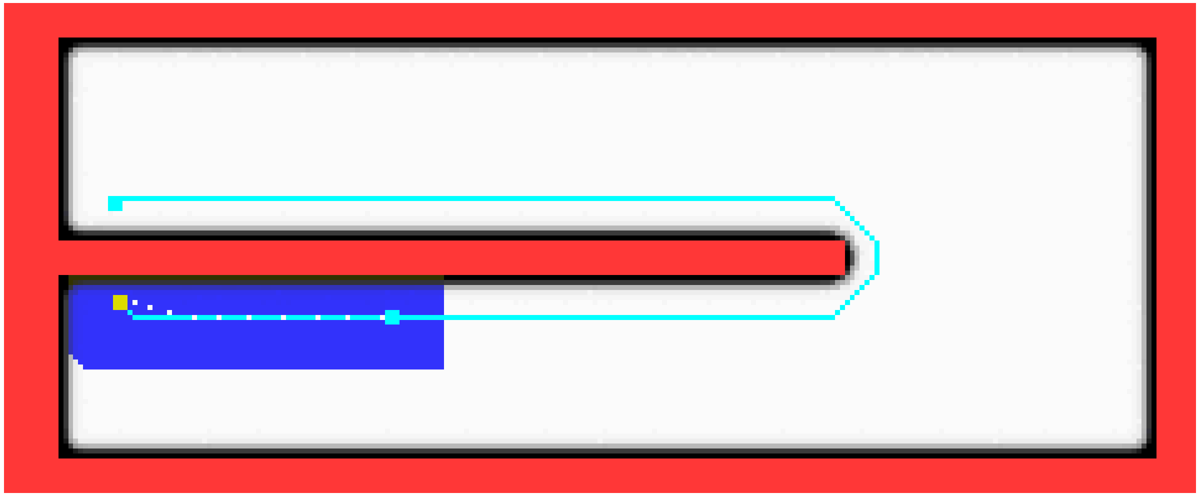


Find a Path in the 5d-Space

- Use A^* in the restricted 5d-space to find a sequence of steering commands to reach the sub-goal.
- To estimate cell costs: perform a deterministic 2d-value iteration within the channel.

Examples





Timeouts

- Steering a robot online requires to set a new steering command every .25 secs.

➔ Abort search after .25 secs.

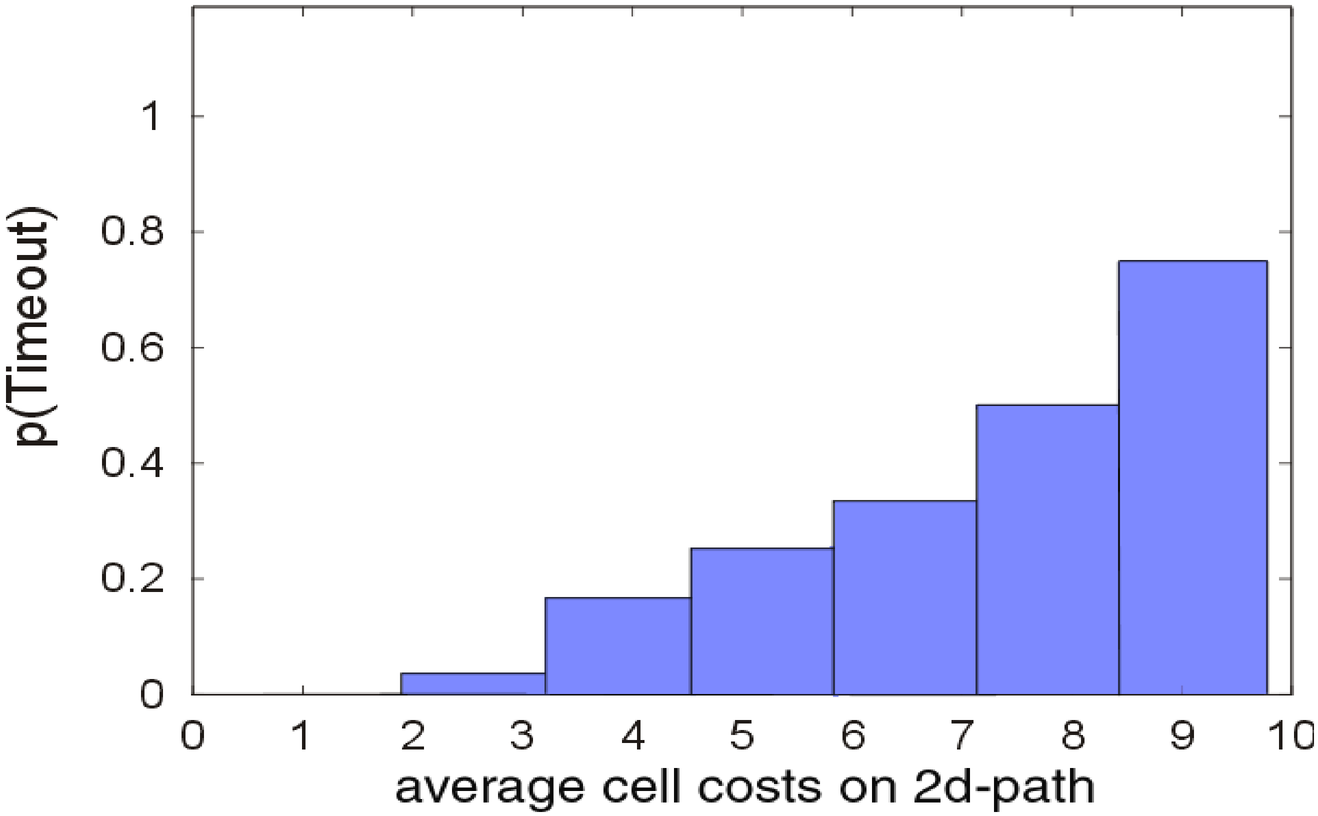
How to find an admissible steering command?

Alternative Steering Command

- Previous trajectory still admissible?
  → **OK**

- If not, drive on the 2d-path or use DWA to find new command.

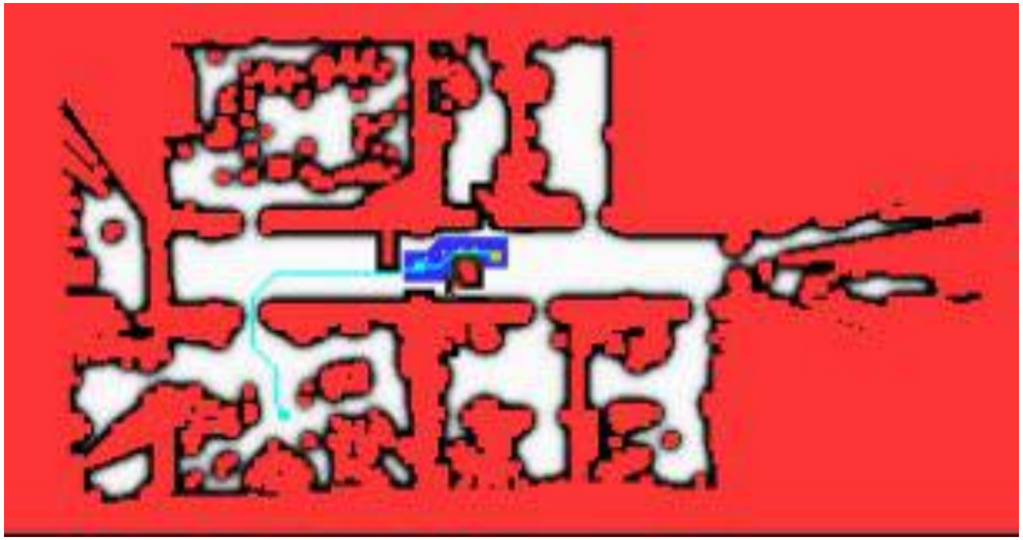
Timeout Avoidance



➔ Reduce the size of the channel if the 2d-path has high cost.

Example

Robot Albert

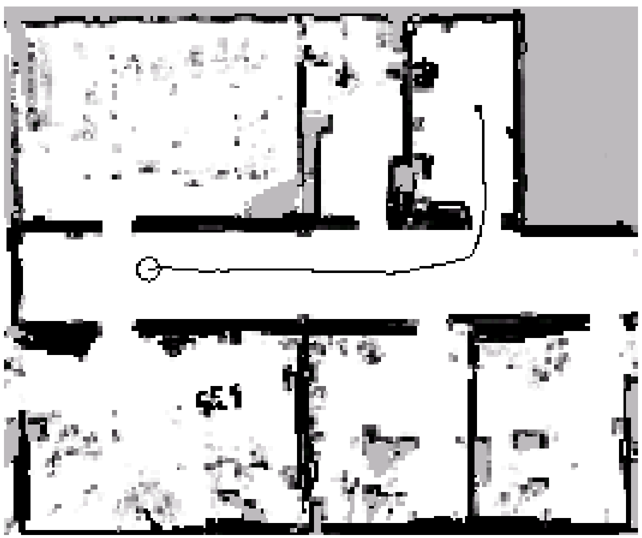


Planning state

Comparison to the DWA (1)

- DWAs often have problems entering narrow passages.

DWA planned path.



5D approach.

Comparison to the DWA (1)

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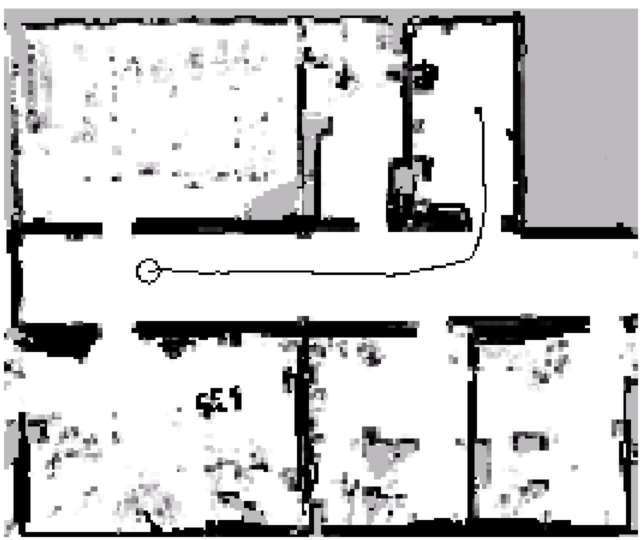
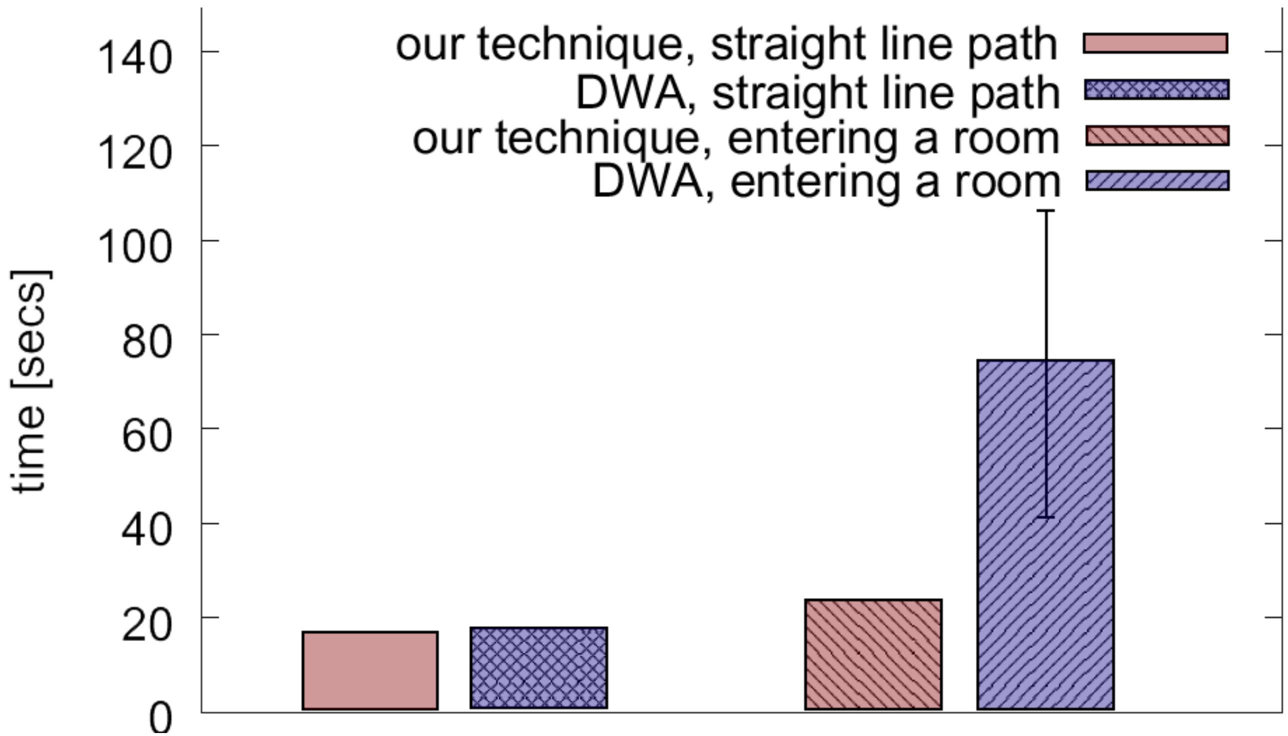
DWA planned path.



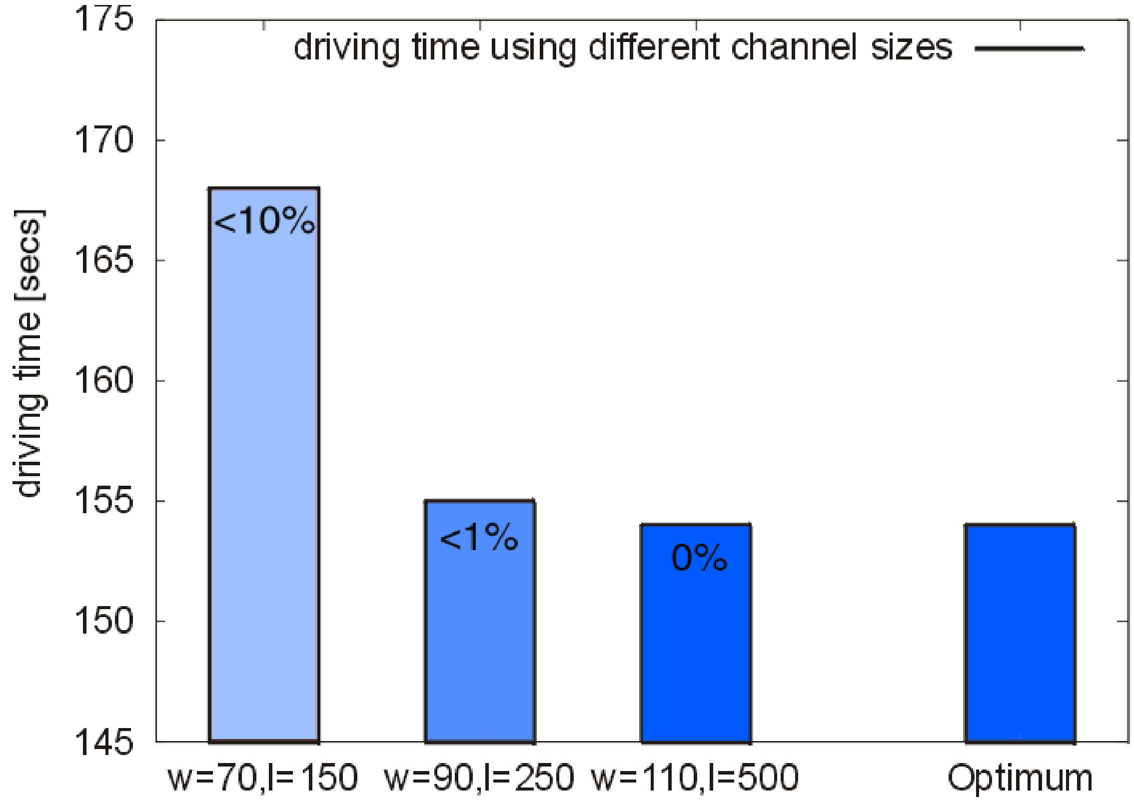
5D approach.

Comparison to the DWA (2)



The presented approach results in significantly faster motion when driving through narrow passages!

Comparison to the Optimum

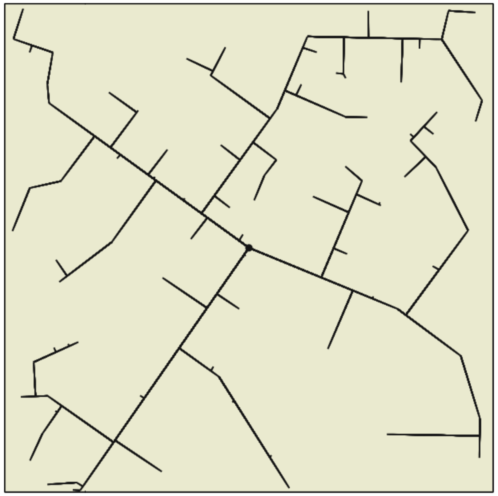


Channel: with length=5m, width=1.1m

Resulting actions are close to the optimal solution.

Rapidly Exploring Random Trees

- **Idea:** aggressively probe and explore the C-space by **expanding incrementally** from an initial configuration q_0
- The explored territory is marked by a **tree rooted at q_0**



45 iterations

2345 iterations

RRTs

The algorithm: Given C and q_0

Algorithm 1: RRT

G.init(q_0)
repeat
    $q_{rand} \rightarrow$ RANDOM_CONFIG($\mathcal{C}$)
    $q_{near} \leftarrow$ NEAREST(G, q_{rand})
    G.add_edge(q_{near}, q_{rand})
until *condition*

Sample from a **bounded region** centered around q_0

E.g. an axis-aligned relative random translation or random rotation

RRTs

- The algorithm

Algorithm 1: RRT

G.init(q_0)
repeat
    $q_{rand} \rightarrow$ RANDOM_CONFIG$(\mathcal{C})$
    $q_{near} \leftarrow$ NEAREST(G, q_{rand})
    G.add_edge(q_{near}, q_{rand})
until *condition*

Finds closest vertex in G using a **distance function**

$$\rho \ : \ \mathcal{C} \times \mathcal{C} \longrightarrow [0, \infty)$$

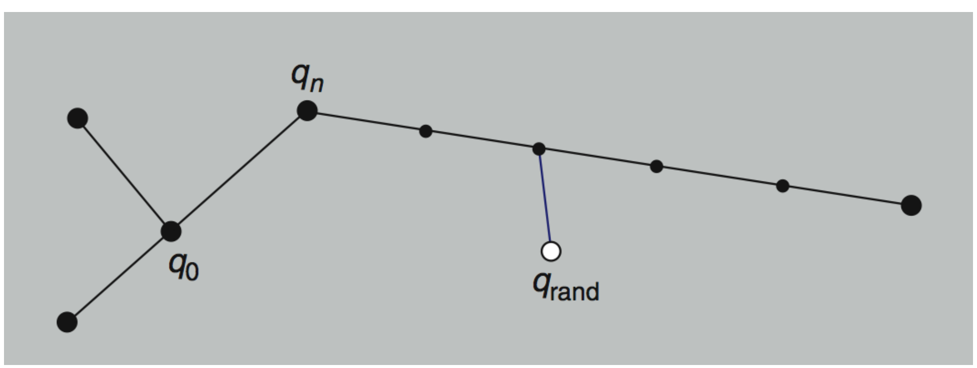
formally a ***metric*** defined on C

RRTs

- The algorithm

Algorithm 1: RRT

G.init(q_0)
repeat
    $q_{rand} \rightarrow$ RANDOM_CONFIG($\mathcal{C}$)
    $q_{near} \leftarrow$ NEAREST(G, q_{rand})
    G.add_edge(q_{near}, q_{rand})
until *condition*



Several stategies to find q_{near} given the closest vertex on G:

- Take closest vertex
- Check intermediate points at regular intervals and split edge at q_{near}

RRTs

- The algorithm

Algorithm 1: RRT

G.init(q_0)
repeat
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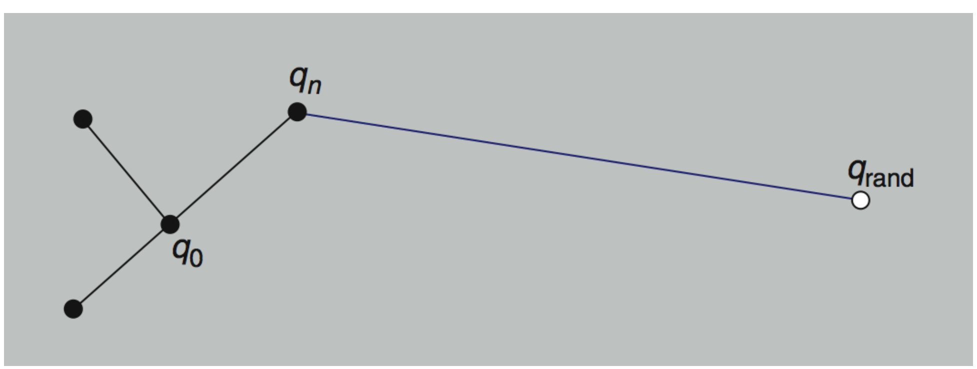
  $q_{near} \leftarrow$ NEAREST(G, q_{rand})
  G.add_edge(q_{near}, q_{rand})
until *condition*

Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}

- No collision: add edge
- Collision: new vertex is q_i, as close as possible to C_{obs}



RRTs

- The algorithm

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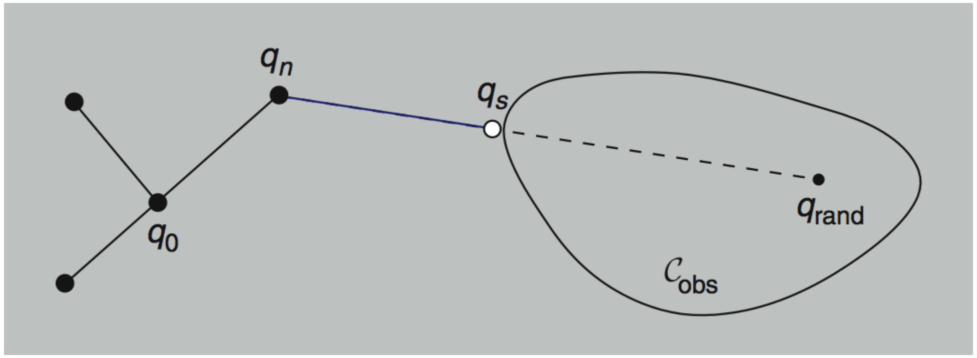


Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}

- No collision: add edge
- Collision: new vertex is q_s, as close as possible to C_{obs}

RRTs

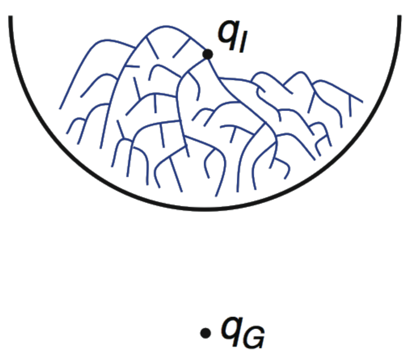
- How to perform path planning with RRTs?

  1. Start RRT at q_I
  2. At every, say, 100th iteration, force $q_{rand} = q_G$
  3. If q_G is reached, problem is solved

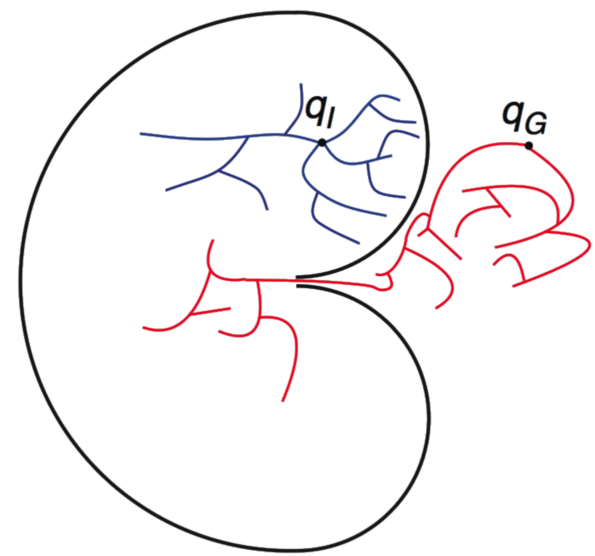
- Why not picking q_G every time?
- This will fail and waste much effort in running into C_{Obs} instead of exploring the space

RRTs

- However, some problems require more effective methods: **bidirectional search**
- Grow **two** RRTs, one from q_I, one from q_G
- In every other step, try to extend each tree towards the newest vertex of the other tree



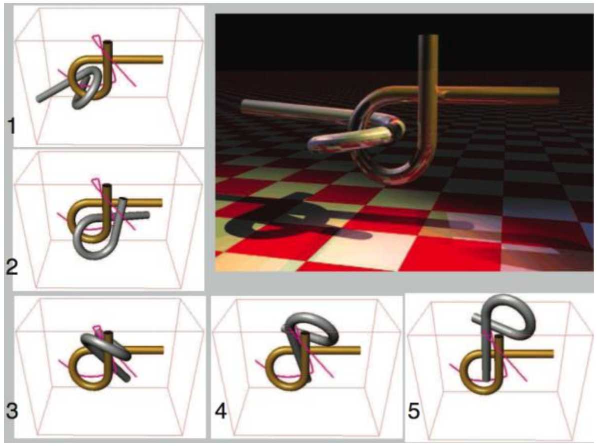
Filling a well



A bug trap

RRTs

- RRTs are popular, many extensions exist: real-time RRTs, anytime RRTs, for dynamic environments etc.
- **Pros:**
  - Balance between greedy search and exploration
  - Easy to implement
- **Cons:**
  - Metric sensivity
  - Unknown rate of convergence



Alpha 1.0 puzzle. Solved with bidirectional RRT

Road Map Planning

- A **road map** is a **graph in $\boldsymbol{C_{free}}$** in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

- Several **planning techniques**
  - Visibility graphs
  - Voronoi diagrams
  - Exact cell decomposition
  - Approximate cell decomposition
  - Randomized road maps

Road Map Planning

- A **road map** is a **graph in** $\boldsymbol{C_{free}}$ in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

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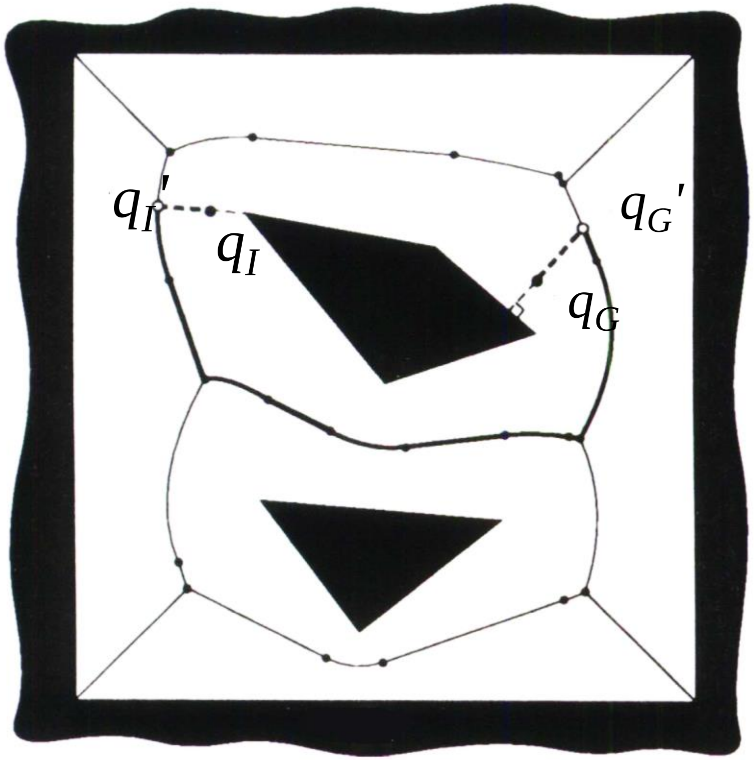
  - Visibility graphs
  - **Voronoi diagrams**
  - Exact cell decomposition
  - Approximate cell decomposition
  - **Randomized road maps**

Generalized Voronoi Diagram

- **Defined** to be the set of points q whose cardinality of the set of boundary points of C_{obs} with the same distance to q is greater than 1
- Let us decipher this definition...
- **Informally:** the place with the same **maximal clearance** from all nearest obstacles



Generalized Voronoi Diagram

- **Formally:**

  Let $\beta = \partial \mathcal{C}_{free}$ be the boundary of C_{free}, and $d(p,q)$ the Euclidian distance between p and q. Then, for all q in C_{free}, let

  $$clearance(q) = \min_{p \in \beta} d(p, q)$$

  be the *clearance* of q, and

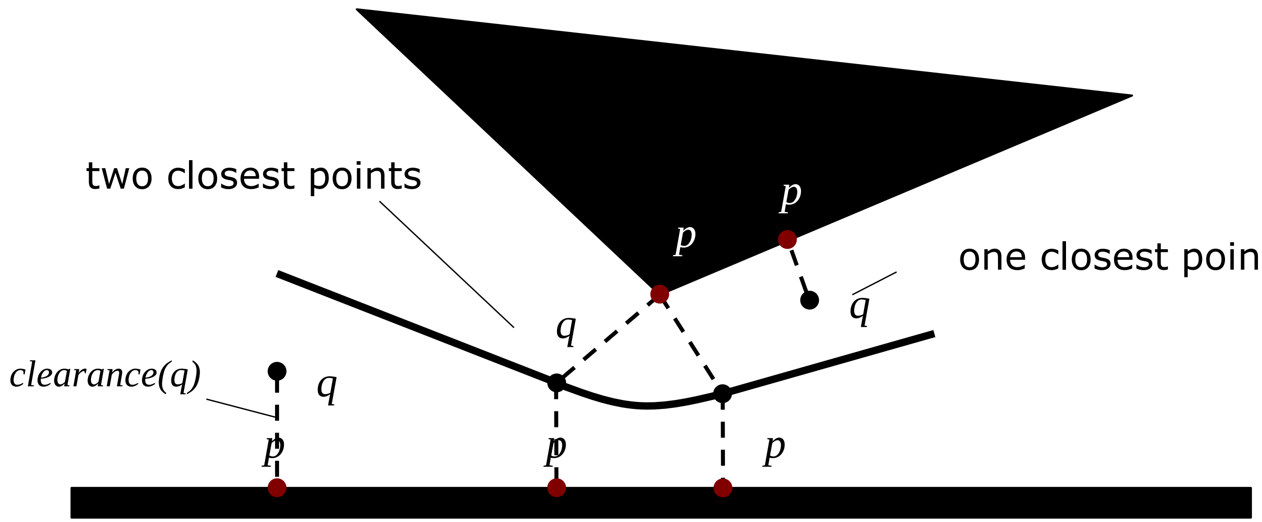
  $$near(q) = \{p \in \beta \mid d(p, q) = clearance(q)\}$$

  the set of "base" points on β with the same clearance to q. The **Voronoi diagram** is then the set of q's with more than one base point p

  $$V(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} \mid |near(q)| > 1\}$$

Generalized Voronoi Diagram

- **Geometrically:**



- For a polygonal C_{obs}, the Voronoi diagram consists of (n) lines and parabolic segments
- Naive algorithm: $O(n^4)$, best: $O(n \log n)$

Voronoi Diagram

- Voronoi diagrams have been well studied for (reactive) **mobile robot** path planning
- Fast methods exist to compute and update the diagram in real-time for low-dim. C's
  - **Pros:** maximize clear-ance is a good idea for an uncertain robot
  - **Cons:** unnatural at-traction to open space, suboptimal paths

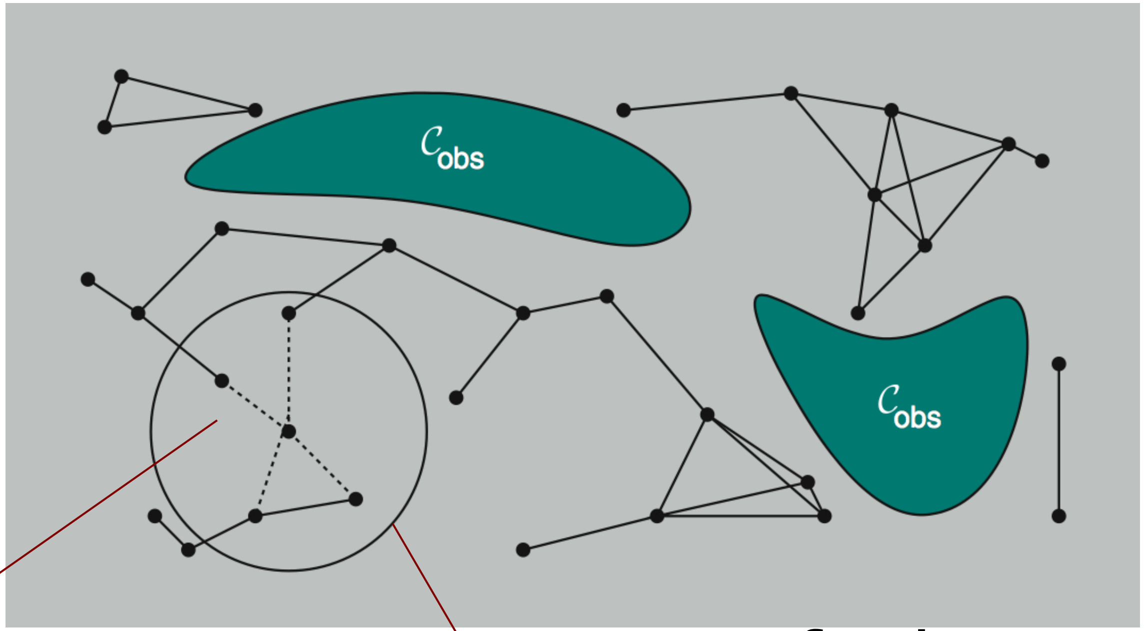
- Needs extensions

Randomized Road Maps

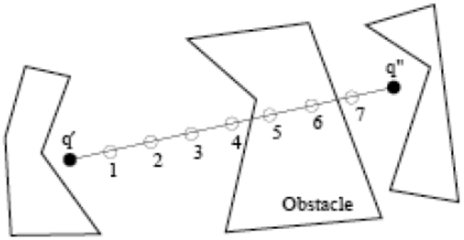
- **Idea:** Take random samples from C, declare them as vertices if in C_{free}, try to connect nearby vertices with local planner
- The **local planner** checks if line-of-sight is collision-free (powerful or simple methods)
- Options for *nearby:* **k-nearest neighbors** or all neighbors within **specified radius**
- Configurations and connections are added to graph until roadmap is **dense enough**

Randomized Road Maps

- Example



specified radius

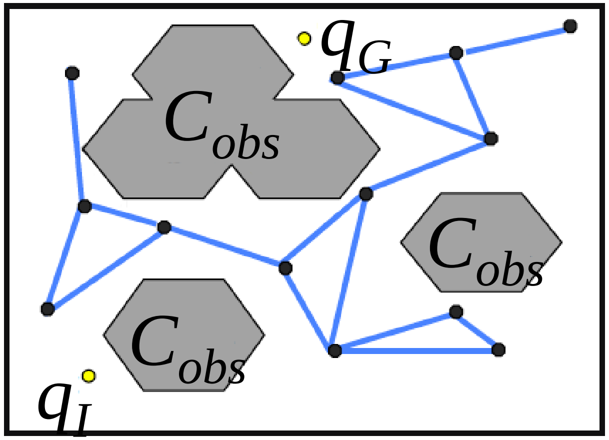


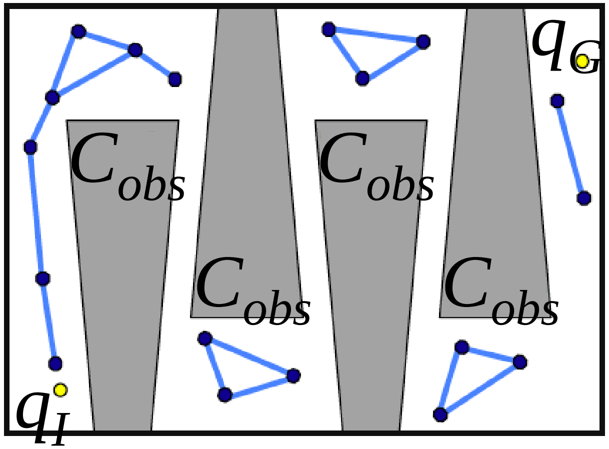
Example local planner

What does "nearby" mean on a manifold? Defining a good metric on C is crucial

Randomized Road Maps

- **Pros:**
  - *Probabilistically complete*
  - Do not construct C-space
  - Apply easily to high dimensional C-spaces
  - Randomized road maps have solved previously unsolved problems





- **Cons:**
  - Do not work well for some problems, narrow passages
  - Not optimal, not complete

Randomized Road Maps

- How to **uniformly sample** C ? This is not at all trivial given its topology
- For example over spaces of rotations: Sampling Euler angles gives more samples near poles, not uniform over $SO(3)$. Use quaternions!
- However, Randomized Road Maps are **powerful, popular** and **many extensions** exist: advanced sampling strategies (e.g. near obstacles), PRMs for deformable objects, closed-chain systems, etc.

From Road Maps to Paths

- All methods discussed so far **construct a road map** (without considering the query pair q_I and q_G)

- Once the investment is made, the **same road map** can be reused for **all** queries (provided world and robot do not change)

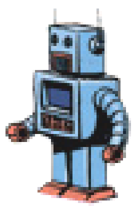
  1. **Find** the cell/vertex that contain/is close to q_I and q_G (not needed for visibility graphs)

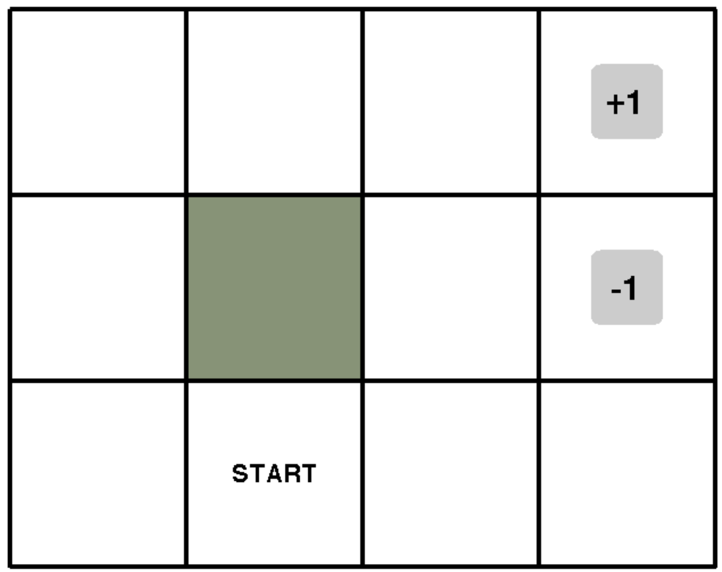
  2. **Connect** q_I and q_G to the road map

  3. **Search** the road map for a path from q_I to q_G

Markov Decision Process

- Consider an agent acting in this environment

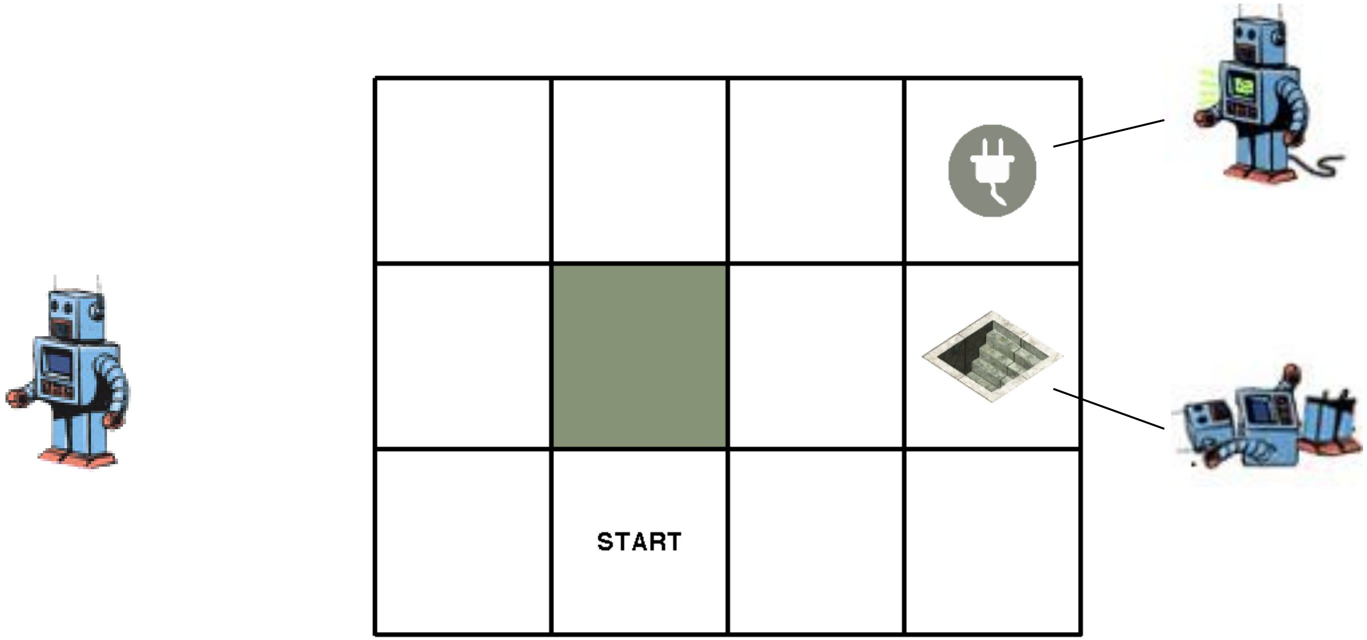




- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

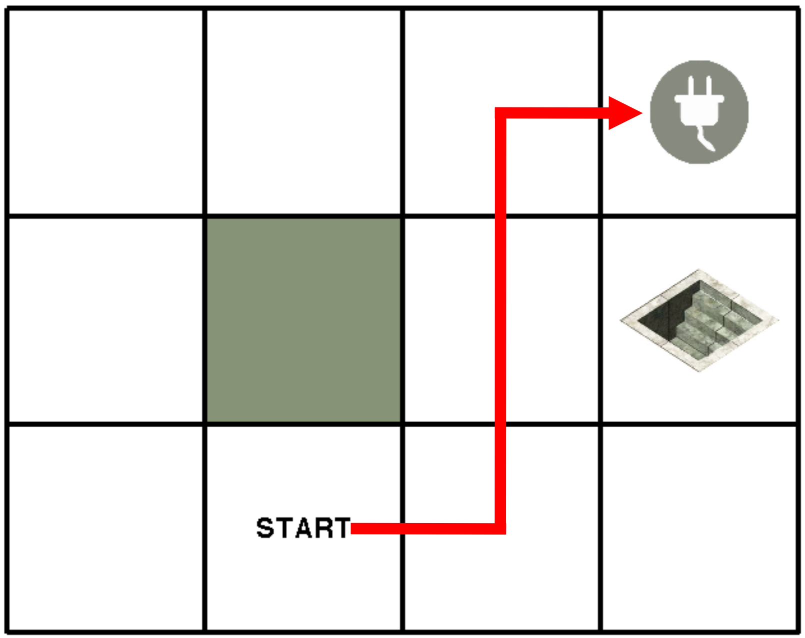
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

- Easy! Use a search algorithm such as A*



- Best solution (shortest path) is the action sequence *[Right, Up, Up, Right]*

What is the problem?

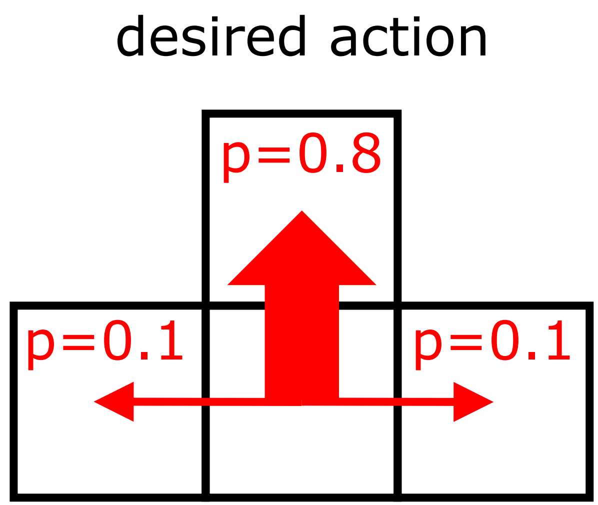
- Consider a non-perfect system
  in which actions are performed with a
  probability less than 1
- What are the best actions for an agent
  under this constraint?

- Example: a mobile robot does not
  exactly perform a desired motion
- Example: human navigation

→ Uncertainty about performing actions!

MDP Example

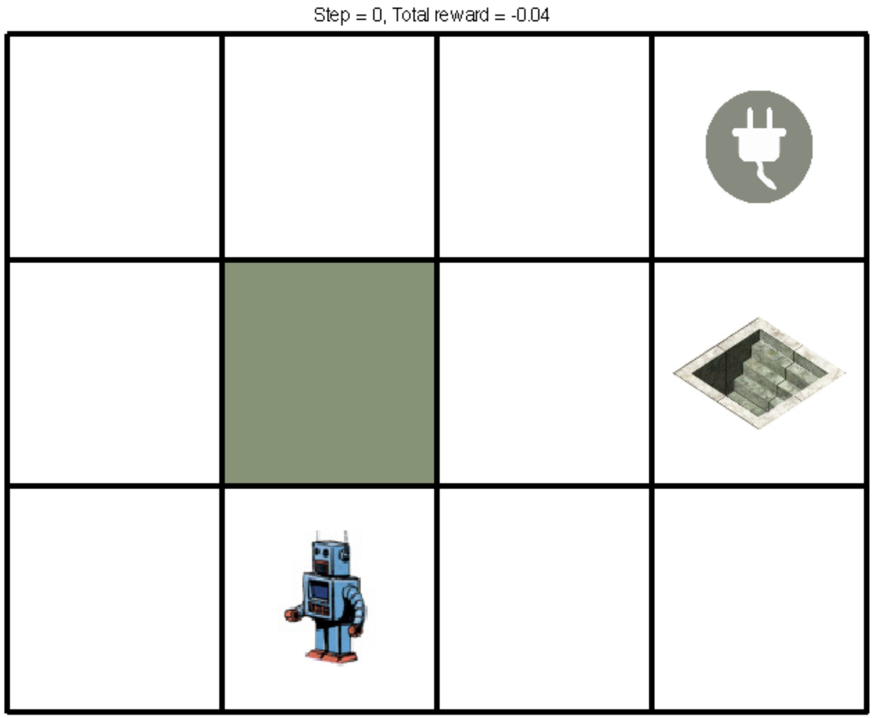
- Consider the **non-deterministic transition model** (N / E / S / W):



- Intended action is executed with p=0.8
- With p=0.1, the agent moves left or right
- Bumping into a wall “reflects” the robot

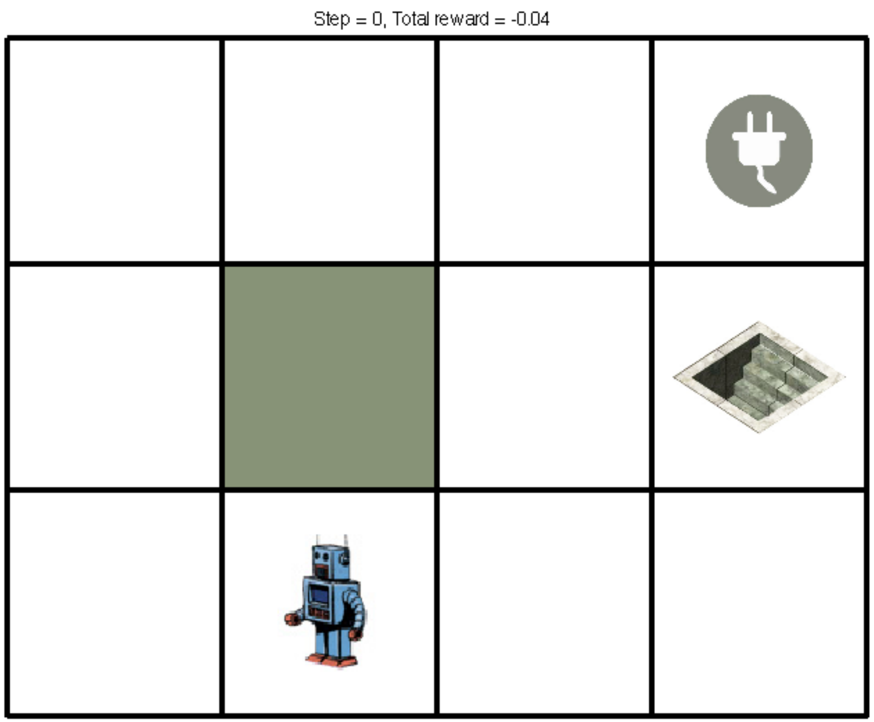
MDP Example

- Executing the **A* plan** in this environment



MDP Example

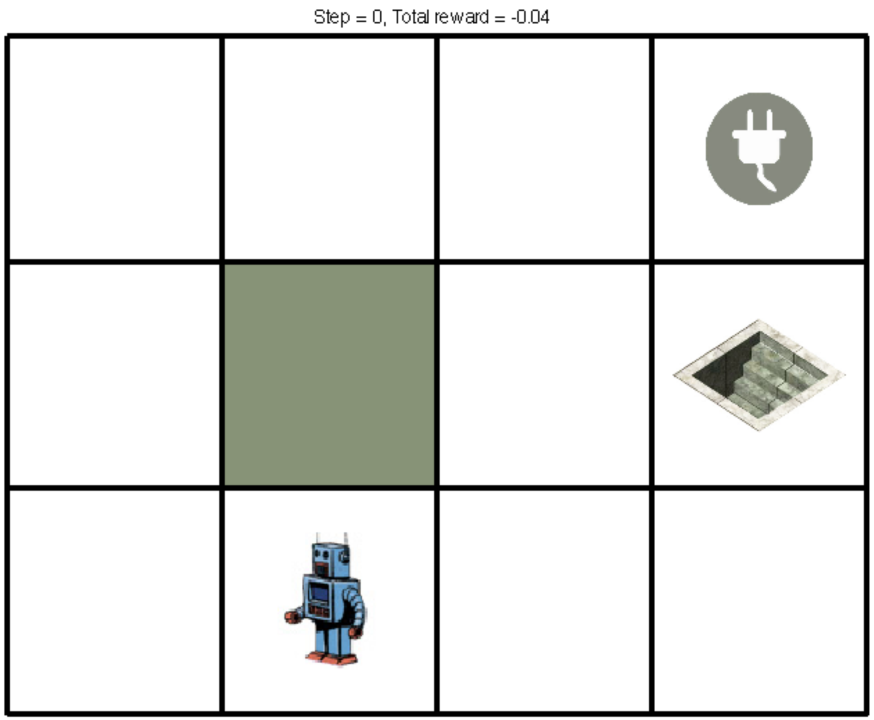
- Executing the **A* plan** in this environment



But: transitions are non-deterministic!

MDP Example

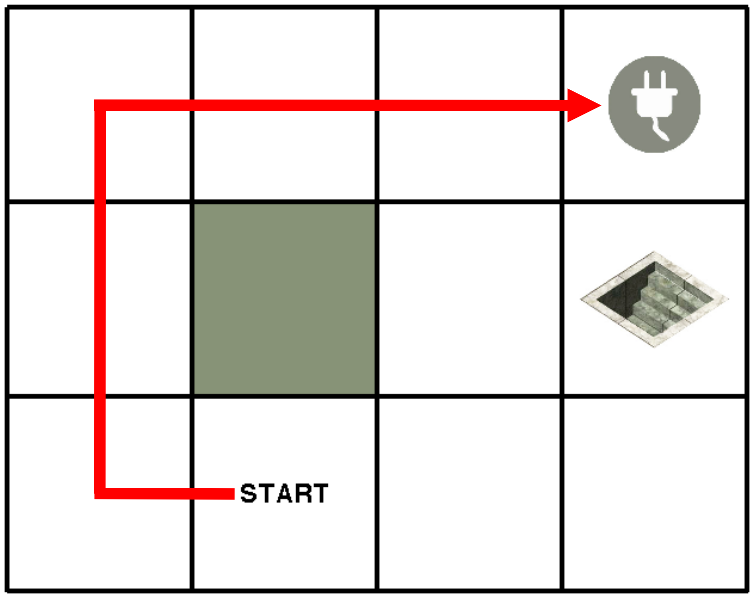
- Executing the **A* plan** in this environment



This will happen sooner or later...

MDP Example

- Use a **longer** path with **lower** probability to end up in cell labelled **-1**



- This path has the **highest overall utility**
- Probability $0.8^6 = 0.2621$

Transition Model

- The probability to reach the next state s' from state s by choosing action a

  $$T(s, a, s')$$

  is called **transition model**

Markov Property:

The transition probabilities from s to s' **depend only on the current state** s and not on the history of earlier states

Reward

- In each state s, the agent receives a **reward** $R(s)$
- The reward may be **positive** or **negative** but must be **bounded**
- This can be generalized to be a function $R(s,a,s')$. Here: consider only $R(s)$, does not change the problem

Reward

- In our example, the reward is **-0.04** in all states (e.g. the cost of motion) except the terminal states (that have rewards **+1**/**-1**)
- A negative reward gives agent an **in-centive to reach the goal quickly**
- Or: “living in this environment is not enjoyable”

-0.04	-0.04	-0.04	+1
-0.04		-0.04	-1
-0.04	-0.04	-0.04	-0.04

MDP Definition

- Given a **sequential decision problem** in a fully observable, stochastic environment with a known Markovian transition model
- Then a **Markov Decision Process** is defined by the components
  - *Set of states:* S
  - *Set of actions:* A
  - *Initial state:* s_0
  - *Transition model:* $T(s, a, s')$
  - *Reward funciton:* $R(s)$

Policy

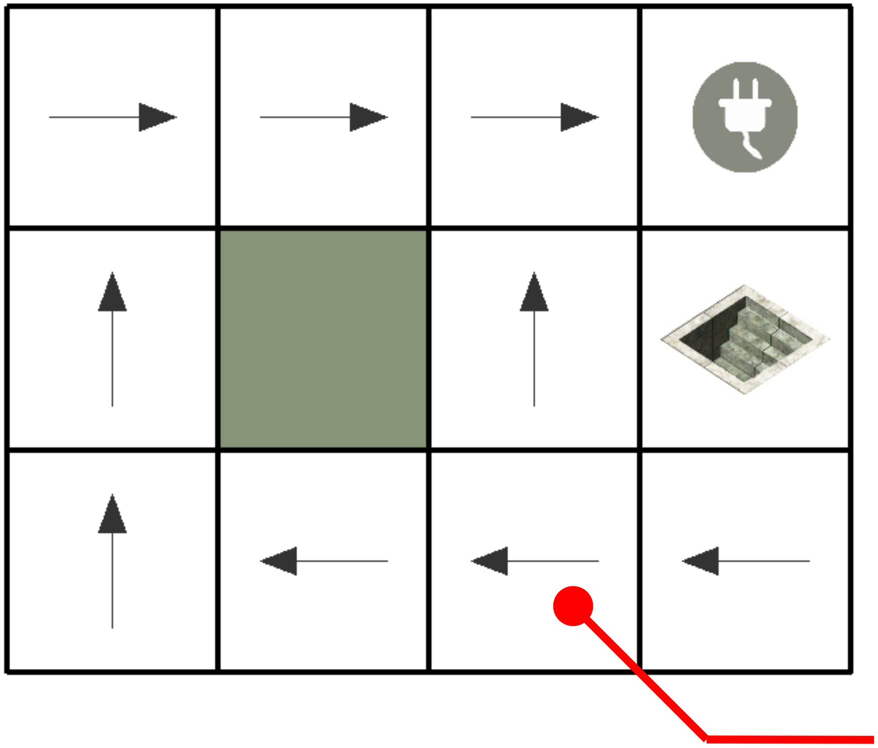
- An MDP solution is called **policy** π
- A policy is a mapping from states to actions

$$policy : States \mapsto Actions$$

- In each state, a policy tells the agent **what to do next**
- Let $\pi(s)$ be the *action* that π specifies for s
- Among the many policies that solve an MDP, the **optimal policy** π* is what we seek. We'll see later what *optimal* means

Policy

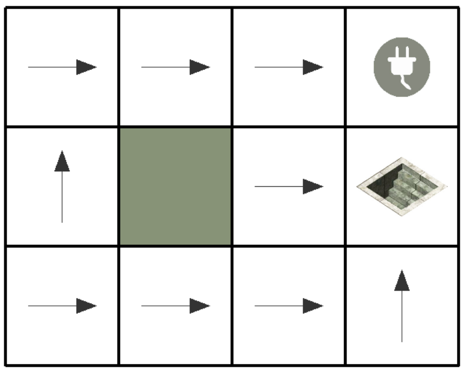
- The optimal policy for our example



Conservative choice
Take long way around as the cost per step of -0.04 is small compared with the penality to fall down the stairs and receive a **-1** reward

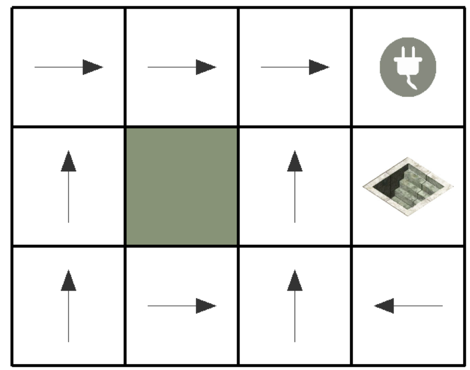
Policy

- When the balance of risk and reward changes, **other policies are optimal**



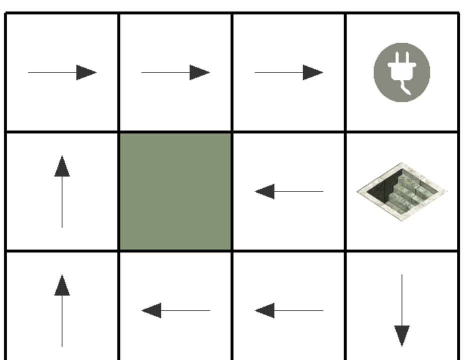
$R < -1.63$

Leave as soon as possible



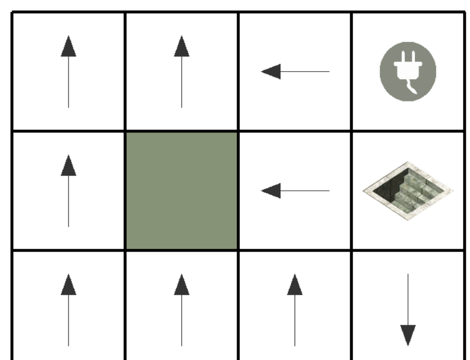
$-0.43 < R < -0.09$

Take shortcut, minor risks



$-0.02 < R < 0$

No risks are taken



$R > 0$

Never leave (inf. #policies)

Utility of a State

- The **utility of a state** $U(s)$ quantifies the **benefit** of a state for the **overall task**
- We first define $U^{\pi}(s)$ to be the **expected utility of all state sequences that start in** s given π

$$U^{\pi}(s) \quad = \quad E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right]$$

- $U(s)$ evaluates (and encapsulates) all possible futures **from** s **onwards**

Utility of a State

- With this definition, we can express $U^{\pi}(s)$ as a **function of its next state** s'

$$
\begin{aligned}
U^{\pi}(s) &= E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right] \\
&= E\left[R(s_0) + R(s_1) + R(s_2) + \ldots \mid \pi, s_0 = s\right] \\
&= E\left[R(s_0) \mid s_0 = s\right] + E\left[R(s_1) + R(s_2) + \ldots \mid \pi\right] \\
&= R(s) + E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s'\right] \\
&= R(s) + U^{\pi}(s')
\end{aligned}
$$

Optimal Policy

- The utility of a state allows us to apply the **Maximum Expected Utility principle** to define the optimal policy π^*

- The **optimal policy** π^* in s chooses the action a that maximizes the expected utility of s (and of s')

$$\pi^*(s) \quad = \quad \underset{a}{\operatorname{argmax}} \; E\left[U^\pi(s)\right]$$

- Expectation taken over all policies

Optimal Policy

- Substituting $U^{\pi}(s)$

$$
\begin{aligned}
\pi^*(s) &= \underset{a}{\operatorname{argmax}}\ E\left[U^{\pi}(s)\right] \\
&= \underset{a}{\operatorname{argmax}}\ E\left[R(s) + U^{\pi}(s')\right] \\
&= \underset{a}{\operatorname{argmax}}\ E\left[R(s)\right] + E\left[U^{\pi}(s')\right] \\
&= \underset{a}{\operatorname{argmax}}\ E\left[U(s')\right] \\
&= \underset{a}{\operatorname{argmax}} \sum_{s'} T(s,a,s')\,U(s')
\end{aligned}
$$

- Recall that $E[X]$ is the weighted average of all possible values that X can take on

Utility of a State

- The **true utility of a state** *U(s)* is then obtained by application of the optimal policy, i.e. $U^{\pi^*}(s) = U(s)$. We find

$$
\begin{aligned}
U(s) &= \max_a E\left[U^\pi(s)\right] \\
&= \max_a E\left[R(s) + U^\pi(s')\right] \\
&= \max_a E\left[R(s)\right] + E\left[U^\pi(s')\right] \\
&= R(s) + \max_a E\left[U(s')\right] \\
&= \underline{R(s) + \max_a \sum_{s'} T(s,a,s')\,U(s')}
\end{aligned}
$$

Utility of a State

- This result is noteworthy:

$$U(s) = R(s) + \max_a \sum_{s'} T(s, a, s')\, U(s')$$

  We have found a direct relationship between the **utility of a state** and the **utility of its neighbors**

- The utility of a state is the immediate reward for that state plus the expected utility of the next state, **provided** the agent chooses the **optimal** action

Bellman Equation

$$\boxed{U(s) = R(s) + \max_a \sum_{s'} T(s,a,s')\,U(s')}$$

- For each state there is a Bellman equation to compute its utility
- There are ***n* states** and ***n* unknowns**
- Solve the system using Linear Algebra?
- No! The max-operator that chooses the optimal action makes the system nonlinear
- We must go for an **iterative approach**

Discounting

We have made a **simplification** on the way:

- The utility of a state sequence is often defined as the sum of **discounted** rewards

$$U^\pi(s) = E\left[\sum_{t=0}^{\infty} \underline{\gamma^t} R(s_t) \mid \pi, s_0 = s\right]$$

  with $0 \,\delta\, \gamma \,\delta\, 1$ being the *discount factor*

- Discounting says that **future** rewards are **less significant** than **current** rewards. This is a natural model for many domains

- The other expressions change accordingly

Separability

We have made an **assumption** on the way:

- Not all utility functions (for state sequences) can be used
- The utility function must have the **property of separability** (a.k.a. stationarity), e.g. additive utility functions:
  $$U([s_0 + s_1 + \ldots + s_n]) = R(s_0) + U([s_1 + \ldots + s_n])$$
- Loosely speaking: the preference between two state sequences is unchanged over different start states

Utility of a State

- The **state utilities** for our example

0.812	0.868	0.918	+1
0.762		0.66	-1
0.705	0.655	0.611	0.388

- Note that utilities are higher closer to the goal as fewer steps are needed to reach it

Iterative Computation

Idea:

- The utility is computed iteratively:

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s')\, U_i(s')$$

- Optimal utility: $U^* = \lim_{t \to \infty} U_t$
- Abort, if change in utility is below a threshold

Dynamic Programming

- The utility function is the basis for "Dynamic Programming"
- Fast solution to compute n-step decision problems
- Naive solution: $O(|A|^n)$
- Dynamic Programming: $O(n\ |A|\ |S|)$
- But: what is the correct value of n?
- If the graph has loops: $n \longrightarrow \infty$

The Value Iteration Algorithm

Algorithm 1: Value Iteration

In: An MDP with
– States and action sets S, A,
– Transition model $T(s, a, s')$,
– Reward function $R(s)$,
– Discount factor γ
Out: The utility of all states U

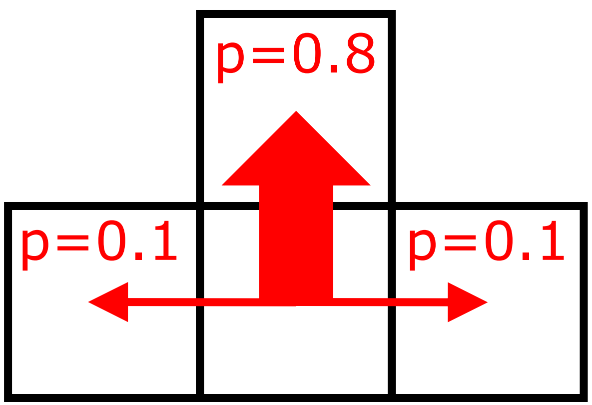
$U' \leftarrow 0$
repeat
  $U \leftarrow U'$
  foreach *state* s *in* S **do**
    $U(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$
  end
until *close-enough*(U, U')
return U

Value Iteration Example

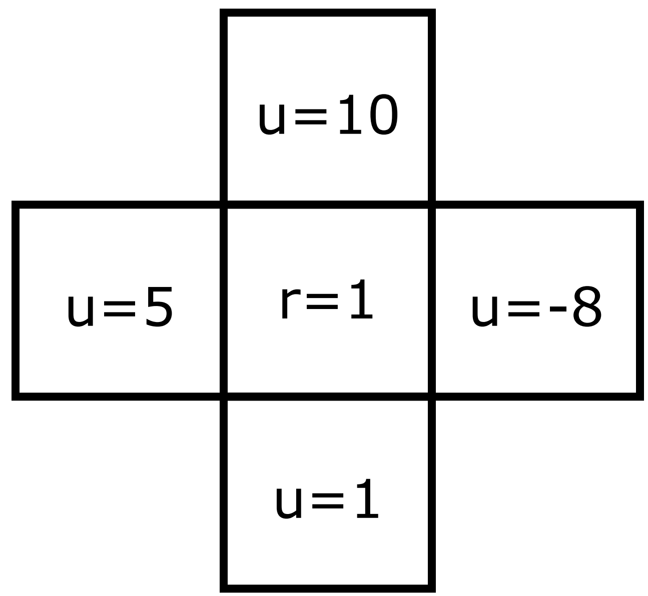
- Calculate utility of the center cell

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s')\, U_i(s')$$

desired action = *Up*



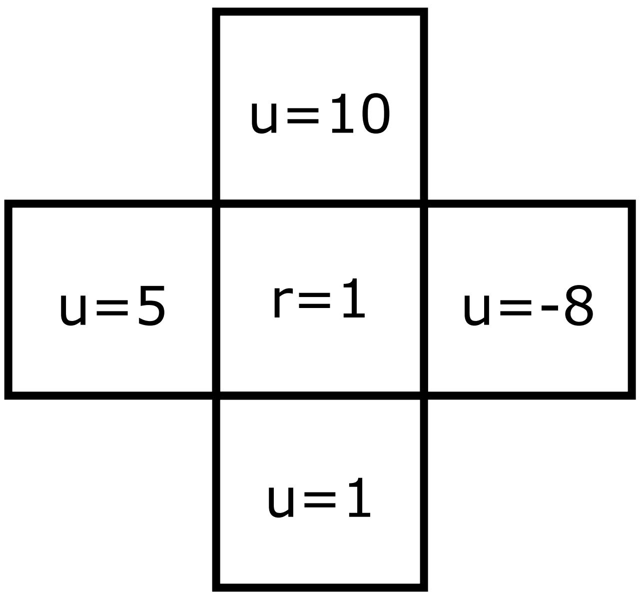
Transition Model



State space
(u=utility, r=reward)

Value Iteration Example

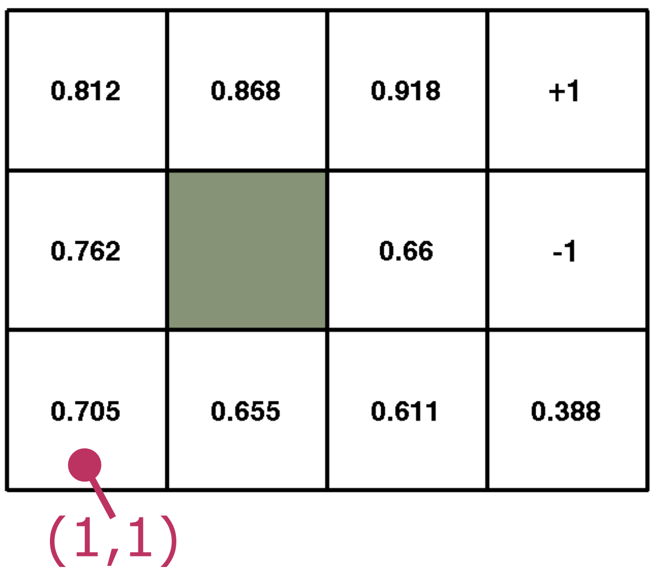
$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s,a,s')\,U_i(s')$$

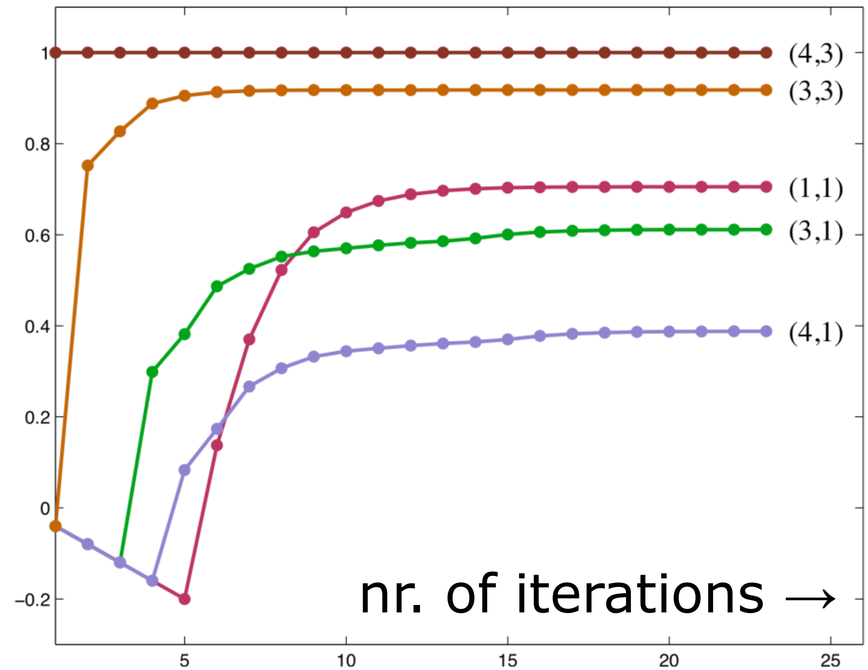


$$\begin{aligned}
&= \mathit{reward} + \mathsf{max}\{\\
&\quad 0.1 \cdot 1 + 0.8 \cdot 5 + 0.1 \cdot 10 \quad (\leftarrow),\\
&\quad 0.1 \cdot 5 + 0.8 \cdot 10 + 0.1 \cdot -8 \quad (\uparrow),\\
&\quad 0.1 \cdot 10 + 0.8 \cdot -8 + 0.1 \cdot 1 \quad (\rightarrow),\\
&\quad 0.1 \cdot -8 + 0.8 \cdot 1 + 0.1 \cdot 5 \quad (\downarrow)\}\\
&= 1 + \mathsf{max}\{5.1\,(\leftarrow), 7.7\,(\uparrow),\\
&\quad -5.3\,(\rightarrow), 0.5\,(\downarrow)\}\\
&= 1 + 7.7\\
&= 8.7
\end{aligned}$$

Value Iteration Example

- In our example





- States far from the goal first accumulate negative rewards until a path is found to the goal

Convergence

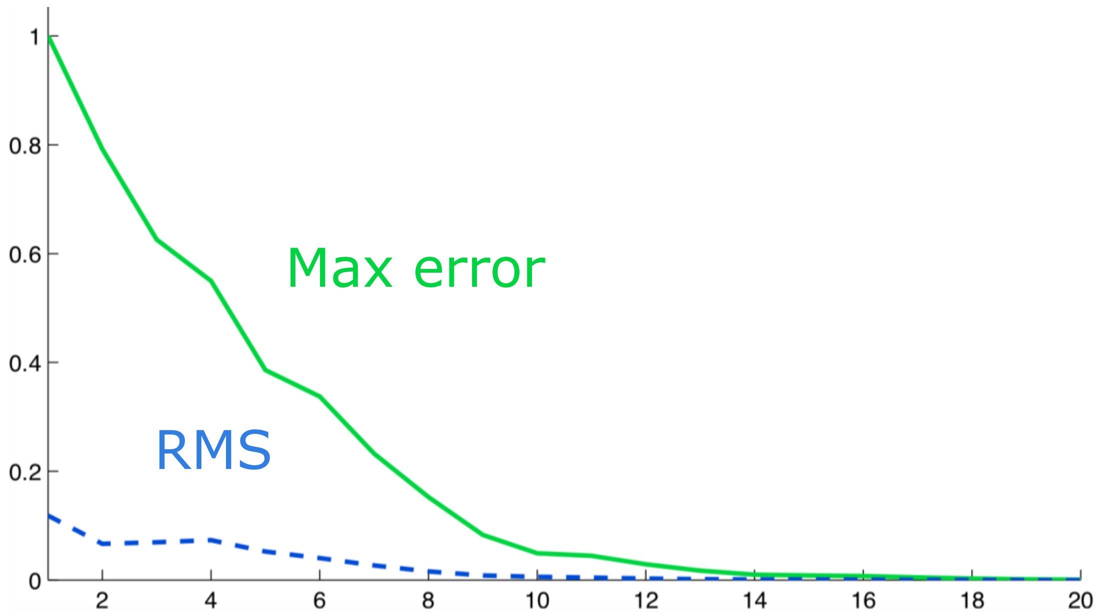
- The condition $close\text{-}enough(U, U')$ in the algorithm can be formulated by

$$RMS = \frac{1}{|S|}\sqrt{\sum_s (U(s) - U'(s))^2}$$

$$RMS(U, U') < \epsilon$$

- Different ways to detect convergence:
  - RMS error: root mean square error
  - Max error: $\|U - U'\| = \max_s |U(s) - U'(s)|$
  - Policy loss

Convergence Example



- What the agent cares about is **policy loss**: How well a policy based on $U_i(s)$ performs
- Policy loss converges much faster (because of the argmax)

Value Iteration

- Value Iteration finds the **optimal solution** to the Markov Decision Problem!
- **Converges** to the **unique solution** of the Bellman equation system
- Initial values for U' are arbitrary
- Proof involves the concept of *contraction*. $\| B\,U_i - B\,U_i' \| \leq \gamma \, \| U_i - U_i' \|$ with B being the Bellman operator (see textbook)
- VI propagates information through the state space by means of **local updates**

Optimal Policy

- How to finally compute the **optimal policy**? Can be easily extracted along the way by

$$\pi^*(s) = \operatorname*{argmax}_a \sum_{s'} T(s,a,s')\, U(s')$$

- **Note:** *U(s)* and *R(s)* are quite different quantities. *R(s)* is the **short-term** reward for being in *s*, whereas *U(s)* is the **long-term** reward **from** *s* **onwards**

Summary

- Robust navigation requires combined path planning & collision avoidance.
- Approaches need to consider robot's kinematic constraints and plans in the velocity space.
- Combination of search and reactive techniques show better results than the pure DWA in a variety of situations.
- Using the 5D-approach the quality of the trajectory scales with the performance of the underlying hardware.
- The resulting paths are often close to the optimal ones.

Summary

- Planning is a complex problem.
- Focus on subset of the configuration space:
  - road maps,
  - grids.
- Sampling algorithms are faster and have a trade-off between optimality and speed.
- Uncertainty in motion leads to the need of Markov Decision Problems.

What's Missing?

- More complex vehicles (e.g., cars).
- Moving obstacles, motion prediction.
- High dimensional spaces.
- Heuristics for improved performances.
- Learning.