Theoretical Computer Science (Bridging Course)

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Exercise Sheet 12

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Exercise 12.1 (Resolution)

Consider the knowledge base $KB = \{A, B \vee E \vee \neg D, K \wedge E \leftrightarrow A \wedge B, \neg C \rightarrow D, E \vee F \rightarrow \neg D\}$. Use resolution to prove that $KB \models A \wedge C$.

Hint: According to *Contradiction Theorem*, $KB \models A \wedge C$ iff $KB \cup \{\neg(A \wedge C)\}$ is unsatisfiable.

Solution: According to the *Contradiction Theorem*, in order to prove that $KB \models A \wedge C$ we can equivalently show, by dint of Resolution, that $KB' = KB \cup \{\neg(A \wedge C)\}$ is unsatisfiable . We first have to convert KB' into clause form:

Formula (and equivalences)	**Clauses**
A	$\{A\}$
$B \vee E \vee \neg D$	$\{B, E, \neg D\}$
$K \wedge E \leftrightarrow A \wedge B$ $\equiv (K \wedge E \rightarrow A \wedge B) \wedge (A \wedge B \rightarrow K \wedge E)$ $\equiv (\neg(K \wedge E) \vee (A \wedge B)) \wedge (\neg(A \wedge B) \vee (K \wedge E))$ $\equiv (\neg K \vee \neg E \vee (A \wedge B)) \wedge (\neg A \vee \neg B \vee (K \wedge E))$ $\equiv (\neg K \vee \neg E \vee A) \wedge (\neg K \vee \neg E \vee B) \wedge$ $(\neg A \vee \neg B \vee K) \wedge (\neg A \vee \neg B \vee E)$	$\{\neg K, \neg E, A\}$ $\{\neg K, \neg E, B\}$ $\{\neg A, \neg B, K\}$ $\{\neg A, \neg B, E\}$
$\neg C \rightarrow D \equiv \neg\neg C \vee D \equiv C \vee D$	$\{C, D\}$
$E \vee F \rightarrow \neg D \equiv \neg(E \vee F) \vee \neg D$ $\equiv (\neg E \wedge \neg F) \vee \neg D \equiv (\neg E \vee \neg D) \wedge (\neg F \vee \neg D)$	$\{\neg E, \neg D\}$ $\{\neg F, \neg D\}$
$\neg(A \wedge C) \equiv \neg A \vee \neg C$	$\{\neg A, \neg C\}$

We want now to derive the empty clause from the set

$$\Delta := \{\{A\}, \{B, E, \neg D\}, \{\neg K, \neg E, A\}, \{\neg K, \neg E, B\}, \{\neg A, \neg B, K\}, \{\neg A, \neg B, E\}, \{C, D\}, \{\neg E, \neg D\}, \{\neg F, \neg D\}, \{\neg A, \neg C\}\}.$$

One possible derivation is the following:

$C_1 = \{A\}$	from Δ
$C_2 = \{\neg A, \neg C\}$	from Δ
$C_3 = \{\neg C\}$	from C_1 and C_2
$C_4 = \{B, E, \neg D\}$	from Δ
$C_5 = \{\neg A, \neg B, E\}$	from Δ
$C_6 = \{\neg A, E, \neg D\}$	from C_4 and C_5
$C_7 = \{E, \neg D\}$	from C_1 and C_6
$C_8 = \{\neg E, \neg D\}$	from Δ
$C_9 = \{\neg D\}$	from C_7 and C_8
$C_{10} = \{C, D\}$	from Δ
$C_{11} = \{C\}$	from C_9 and C_{10}
$C_{12} = \square$	from C_3 and C_{11}

The statement is finally proved.

Exercise 12.2 (Predicate Logic,Terminology)
Classify the following expressions as *terms*, *ground terms*, *atoms* and *formulae*. If there is more than one possibility for an expression, please list them all. In the expressions, a and b are constant symbols, x and y are variable symbols, f and g are function symbols, and P and Q are relation symbols.

(a) $\mathrm{P}(x, y)$

(b) $\mathrm{f}(\mathrm{a}, \mathrm{b})$

(c) $\mathcal{I} \models \mathrm{P}(\mathrm{a}, \mathrm{f}(\mathrm{b}))$

(d) $\mathcal{I}, \alpha \models \mathrm{P}(\mathrm{a}, \mathrm{f}(x))$

(e) $\mathrm{f}(\mathrm{g}(x), \mathrm{b})$

(f) $\mathrm{Q}(x)$ is satisfiable.

(g) $\exists x(\mathrm{P}(x, y) \wedge \mathrm{Q}(x)) \vee \mathrm{P}(y, x)$

(h) $\forall x(\exists y(\mathrm{P}(x, y) \wedge \mathrm{Q}(x)) \vee \mathrm{P}(x, y))$

(i) $\forall x \forall y(\mathrm{P}(x, y) \wedge \mathrm{Q}(x) \vee \mathrm{P}(\mathrm{f}(y), x))$

(j) $\mathrm{Q}(x) \vee \mathrm{P}(x, y) \equiv \mathrm{P}(x, y) \vee \mathrm{Q}(x)$

Solution:

- terms: b, e
- ground terms: b
- atoms: a
- formulae: a, g, h, i

Exercise 12.3 (**Extra**, Predicate Logic, Interpretation)
Consider the following set of formulae:

$$KB = \left\{ \begin{array}{l} \forall x \neg \mathrm{P}(x, x) \\ \forall x \forall y \forall z((\mathrm{P}(x, y) \wedge \mathrm{P}(y, z)) \rightarrow \mathrm{P}(x, z)) \\ \forall x \forall y(\mathrm{P}(x, y) \vee (x = y) \vee \mathrm{P}(y, x)) \end{array} \right\}$$

- Specify an interpretation $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D} = \{d_1, \ldots, d_4\}$ and prove that $\mathcal{I} \models KB$ (i.e., $\mathcal{I} \models \varphi$ for all $\varphi \in KB$). Why is it not necessary to specify a variable assignment α to state a model of KB?

  Solution: Consider the interpretation $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with

  $$\mathrm{P}^{\mathcal{I}} = \{(d_1, d_2), (d_1, d_3), (d_1, d_4), (d_2, d_3), (d_2, d_4), (d_3, d_4)\}$$

  Since all variables are bound, we don't need to specify a variable assignment, but the following needs to hold for all variable assignments α:

  In order to prove that $\mathcal{I}, \alpha \models KB$, we have to show that $\mathcal{I}, \alpha \models \varphi$ for each $\varphi \in KB$.

$$
\begin{aligned}
\mathcal{I},\alpha \models \forall x \neg \mathrm{P}(x,x) &\text{ if } \mathcal{I},\alpha[x:=d] \models \neg\mathrm{P}(x,x) \text{ for all } d \in \mathcal{D}\\
&\text{ if } \mathcal{I},\alpha[x:=d] \not\models \mathrm{P}(x,x) \text{ for all } d \in \mathcal{D}\\
&\text{ if } (d,d) \notin \mathrm{P}^{\mathcal{I}} \text{ for all } d \in \mathcal{D},\\
&\text{ which obviously holds.}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I},\alpha \models &\forall x \forall y \forall z((\mathrm{P}(x,y) \wedge \mathrm{P}(y,z)) \to \mathrm{P}(x,z))\\
&\text{if } \mathcal{I},\alpha[x:=d] \models \forall y \forall z((\mathrm{P}(x,y) \wedge \mathrm{P}(y,z)) \to \mathrm{P}(x,z))\\
&\quad \text{for all } d \in \mathcal{D}\\
&\text{if } \mathcal{I},\alpha[x:=d, y:=d'] \models \forall z((\mathrm{P}(x,y) \wedge \mathrm{P}(y,z)) \to \mathrm{P}(x,z))\\
&\quad \text{for all } d,d' \in \mathcal{D}\\
&\text{if } \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models (\mathrm{P}(x,y) \wedge \mathrm{P}(y,z)) \to \mathrm{P}(x,z)\\
&\quad \text{for all } d,d',d'' \in \mathcal{D}\\
&\text{if (if } \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models \mathrm{P}(x,y) \wedge \mathrm{P}(y,z) \text{, then}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models \mathrm{P}(x,z) \text{ for all } d,d',d'' \in \mathcal{D})\\
&\text{if (if } \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models \mathrm{P}(x,y) \text{ and}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models \mathrm{P}(y,z) \text{, then}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d', z/d''] \models \mathrm{P}(x,z) \text{ for all } d,d',d'' \in \mathcal{D})\\
&\text{if (if } (d,d') \in \mathrm{P}^{\mathcal{I}} \text{ and } (d',d'') \in \mathrm{P}^{\mathcal{I}} \text{, then } (d,d'') \in \mathrm{P}^{\mathcal{I}}\\
&\quad \text{for all } d,d',d'' \in \mathcal{D}),\\
&\text{which is easy to see for } \mathrm{P}^{\mathcal{I}}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I},\alpha \models &\forall x \forall y (\mathrm{P}(x,y) \vee x = y \vee \mathrm{P}(y,x))\\
&\text{if for all } d \in \mathcal{D}\\
&\quad \mathcal{I},\alpha[x:=d] \models \forall y (\mathrm{P}(x,y) \vee x = y \vee \mathrm{P}(y,x))\\
&\text{if for all } d,d' \in \mathcal{D}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d'] \models \mathrm{P}(x,y) \vee x = y \vee \mathrm{P}(y,x)\\
&\text{if for all } d,d' \in \mathcal{D}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d'] \models \mathrm{P}(x,y) \text{ or}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d'] \models x = y \text{ or}\\
&\quad \mathcal{I},\alpha[x:=d, y:=d'] \models \mathrm{P}(y,x)\\
&\text{if for all } d,d'' \in \mathcal{D}\\
&\quad (d,d') \in \mathrm{P}^{\mathcal{I}} \text{ or } d = d' \text{ or } (d',d) \in \mathrm{P}^{\mathcal{I}},\\
&\text{which again is easy to see for } \mathrm{P}^{\mathcal{I}}.
\end{aligned}
$$