Theoretical Computer Science (Bridging Course)

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Exercise Sheet 1

Exercise 1.1 (Graphs)

Let $\mathcal{G} := (G, E)$ be an undirected graph. We define the *degree* of a vertex $g \in G$ to be the number of edges incident to g. We say that $\mathcal{G}$ is k**-regular** ($k \geq 0$) if each vertex $g \in G$ has degree k, or, equivalently, if every vertex is directly connected by an edge to exactly k other vertices.

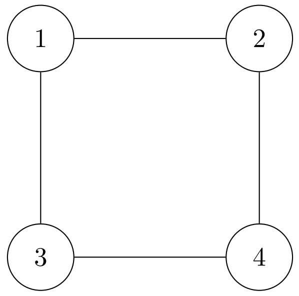
Prove that for every integer $k \geq 2$ there exists a k-regular graph $\mathcal{G}_k := (G_k, E_k)$ so that $|G_k| = 2k$ and diameter of $\mathcal{G}_k$ is 2.

We recall that the *diameter* is the longest shortest path between any two vertices of the graph.

Proof
We will show the statement in two ways: by construction and by induction.

Induction

– *Base*: $k = 2$, it is enough to provide an example of a graph with 4 vertices satisfying the above conditions. An example of such graph is the following:



It is apparent that each vertex has degree 2 (exactly 2 neighbours) and diameter is 2 since any node is connected to any other by a path of length at most 2 and the paths from 1 to 4 are exactly of length 2.

- *Induction hypothesis*: for $k - 1$ there exists a $k - 1$-regular graph with $2k - 2$ vertices and whose diameter is 2.

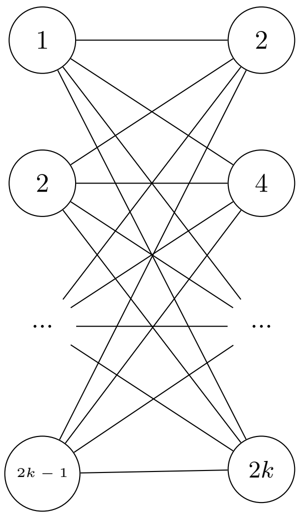
- *Induction step* Let $\mathcal{G}_{k-1} := (G_{k-1}, E_{k-1})$ one graph that satisfies the above conditions. We define a graph $\mathcal{G}_k := (G_k, E_k)$ as follows: we add two vertex to $\mathcal{G}_{k-1}$, formally $G_k = G_{k-1} \cup \{g_1, g_2\}$, furthermore, since the number of nodes of $\mathcal{G}_{k-1}$ is even, we connect half of its node with g_1 and the other half with g_2. Moreover we connect g_1 directly with g_2. As a consequence:

  – $|G_k| = |G_{k-1}| + 2 = 2k$

  – $\mathcal{G}_k$ is k-regular. To understand this, we need to observe that if $g \in G_{k-1} \subset G_k$ than g has exactly $k - 1 + 1$ neighbours ($k - 1$ comes from the $k - 1$-regularity, the $+1$ bit is due to the fact that the node is now connected either to g_1 or to g_2). If $g \notin G_{k-1}$, then $g = g_1$ or $g = g_2$. Say $g = g_1$, then g_1 has again $k - 1 + 1$ neighbours (the $k - 1$ bit comes from the fact that it has been connected to half of the nodes of G_{k-1}, the $+1$ part is g_2). Similarly for g_2.

– We need to show that diameter of $\mathcal{G}_k$ is 2 but this is trivial: g_1 and g_2 are directly connected as well as g_1 is directly connected to half of the nodes of the other graphs. If we want to reach a node g in the other half, we can just create a path that starts in g_2, goes to g_1 and then to the node g. So length 2. Similar argument for g_2. So the diameter do not increase and cannot decrease because no connection amongst the node of G_{k-1} have been added.

Construction: consider the following graph $\mathcal{G}_k := (G_k, E_k)$:



Then it is easy to see that

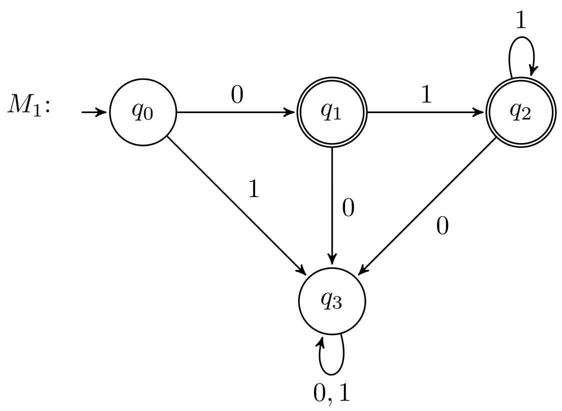
– $|G_k| = 2k$

– Each node in the left column is connected only to the nodes in the right column, and to all of them. Similarly for the right column. So each vertex has exactly k neighbours. Consequently $\mathcal{G}_k$ is k-regular.

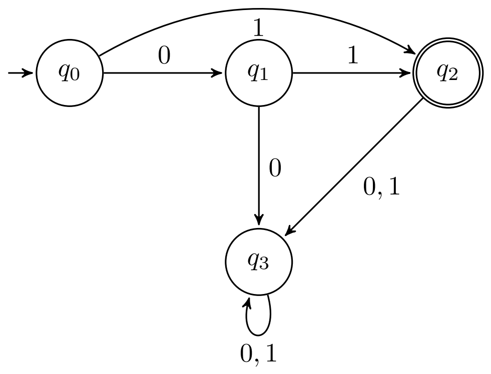
– Any node of the right column is connected by a path of length 1 to any other node in the left column. Furthermore, nodes that lie in the same column are not connected one another, but each couple of nodes are linked by a path of length at most 2, from which we infer that diameter of $\mathcal{G}_k$ is exactly 2.

Exercise 1.2 (DFA)

Consider the following two DFAs (deterministic finite automata) with $\Sigma = \{0, 1\}$:



M_2:



(a) What languages (L_1 and L_2) do these two automata individually recognize?

Solution
$L_1 = \{w \mid w \text{ starts with 0 and is followed by 0 or more } 1s\}$
$L_2 = \{01, 1\}$

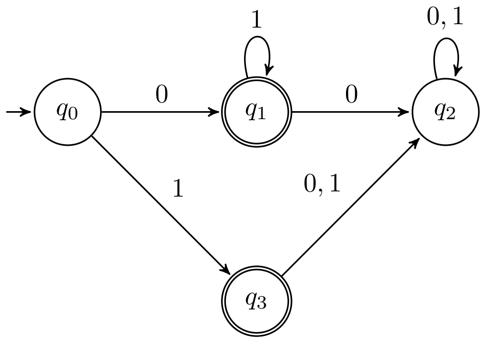
(b) Give the formal definition for M_1.

Solution
$M_1 = (Q, \Sigma, \delta, q_0, F)$ with $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{0, 1\}$, $F = \{q_1, q_2\}$ and δ being given in the table

δ	0	1
q_0	q_1	q_3
q_1	q_3	q_2
q_2	q_3	q_2
q_3	q_3	q_3

(c) Show that $L_1 \cup L_2$ is also a regular language, by constructing **one** DFA. Please hand in a **high quality** diagram.

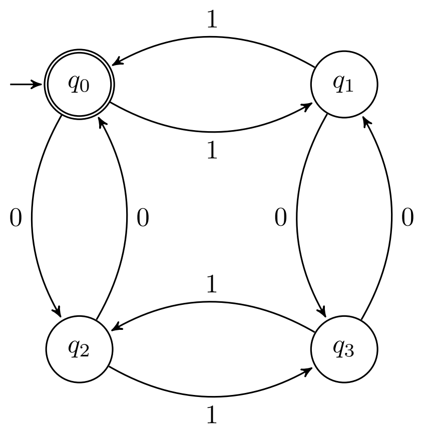
Solution
(It can be constructed using the method/hint from the lecture. But, just for simplicity, the following recognizes $L_1 \cup L_2$ as well.)



Exercise 1.3 (DFA)

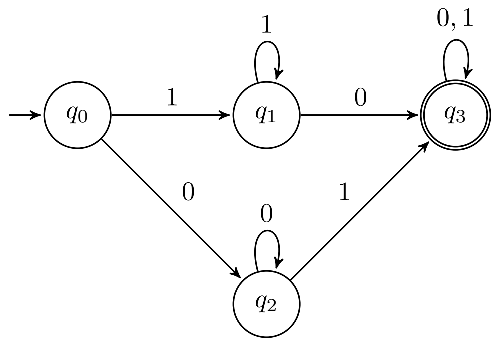
(a) Construct a DFA that recognizes the language L with an alphabet $\Sigma = \{0, 1\}$, where $L = \{w \mid w \text{ has both an even number of 0's and an even number of 1's}\}$

Solution



(b) Give the state diagram for a DFA accepting the language $L = \{w \mid w \text{ starts with 1 and contains 10 or starts with 0 and contains the 01}\}$ The alphabet is $\Sigma = \{0, 1\}$.

Solution



Exercise 1.4 (Regular Language)

In this exercise we want to prove that regular languages are closed under intersection and under complement. The intersection of two languages is defined as $L_1 \cap L_2$. The complement of a language is defined as the set of all words in Σ^* which are not in L, i.e. $\bar{L} = \Sigma^* \setminus L$ (Σ^* is the set of all words/strings over Σ).

Let L and L' be regular languages that are recognized by DFAs $M = (Q, \Sigma, \delta, q_0, F)$ and $M' = (Q', \Sigma', \delta', q_0', F')$, respectively.

(a) Show that the regular languages are closed under *intersection*, i.e. give a finite automaton that recognizes $L \cap L'$.

Solution
We provide a suitable DFA $\overline{M} := (\overline{Q}, \overline{\Sigma}, \overline{\delta}, \overline{q_0}, \overline{F})$ so that $L(\overline{M}) = L \cap L'$. Without loss of generality we can suppose that $\Sigma = \Sigma'$. Indeed, the general case can be proven by setting $\overline{\Sigma} := \Sigma \cap \Sigma'$ and minor rearrangements.
Let $\overline{M}$ be defined as follows:

- $\overline{Q} := Q \times Q'$.
- $\overline{\Sigma} := \Sigma \; (= \Sigma')$.
- $\overline{\delta}$ so that $\overline{\delta}((q, q'), a) := (\delta(q, a), \delta'(q', a))$.
- $\overline{q_0} := (q_0, q_0')$.
- $\overline{F} := F \times F'$.

We need to prove that $w \in L \cap L' \Leftrightarrow w \in L(\overline{M})$.

$\Rightarrow$) Let $w = w_0 \cdots w_{n-1} \in L \cap L'$, then, w is accepted by both M and M'. Thus, by definition of computation it is apparent that there exist two sequences of states $(q_0, q_1, ..., q_n) \subseteq Q$ and $(q_0', q_1', ..., q_n') \subseteq Q'$ so that $q_{i+1} = \delta(q_i, w_i)$ $(i = 0, ..., n-1)$ and $q_n \in F$ as well as $q_{i+1}' = \delta'(q_i', w_i)$ $(i = 0, ..., n-1)$ and $q_n' \in F'$. Applying w to $\overline{M}$, it is easy to see that

$$\overline{\delta}((q_i, q_{i+1}'), w_i) = (q_{i+1}, q_{i+1}'), \; (i = 0, ..., n-1)$$
$$(q_n, q_{n'}) \in F \times F'.$$

That is, the computations ends in an accept state for the DFA $\overline{M}$.

$\Leftarrow$) Let $\overline{w} = \overline{w}_0 \cdots \overline{w}_{n-1} \in L(\overline{M})$. By definition of computation, there exists a sequence of states $(\overline{q_0}, ..., \overline{q_n}) \in \overline{Q}$ so that $\overline{\delta}(\overline{q_i}, \overline{w}_i) = \overline{q_{i+1}}$ and $\overline{q_n} \in \overline{F}$. Consequently, setting $\overline{q_i} := (q_i, q_i')$ and by definition of the transition function $\overline{\delta}$, the following relations hold

$$\begin{aligned} &\delta(q_i, \overline{w}_i) = q_{i+1}, && \delta(q_i', \overline{w}_i) = q_{i+1}', \\ &q_n \in F && q_n' \in F'. \end{aligned}$$

Thus, $\overline{w}$ is accepted by both L and L'. That is $\overline{w} \in L \cap L'$.

(b) Show that the regular languages are closed under *complement*, i.e. give a finite automaton that recognizes $\bar{L}$.

Solution
We define the DFA $M_{\bar{L}} := (Q, \Sigma, \delta, q_0, Q \setminus F)$. As in part (a), we need to prove that $w \in L(M_{\bar{L}}) \Leftrightarrow w \notin L(M)$. However, it is easy to see that whenever a computation ends in a accept state for M, then it cannot be accepted by $M_{\bar{L}}$ as a state is an accept state for M if and only it's not an accept state for $M_{\bar{L}}$. With a similar argument it can be shown that the opposite statement holds too.