Introduction to Mobile Robotics

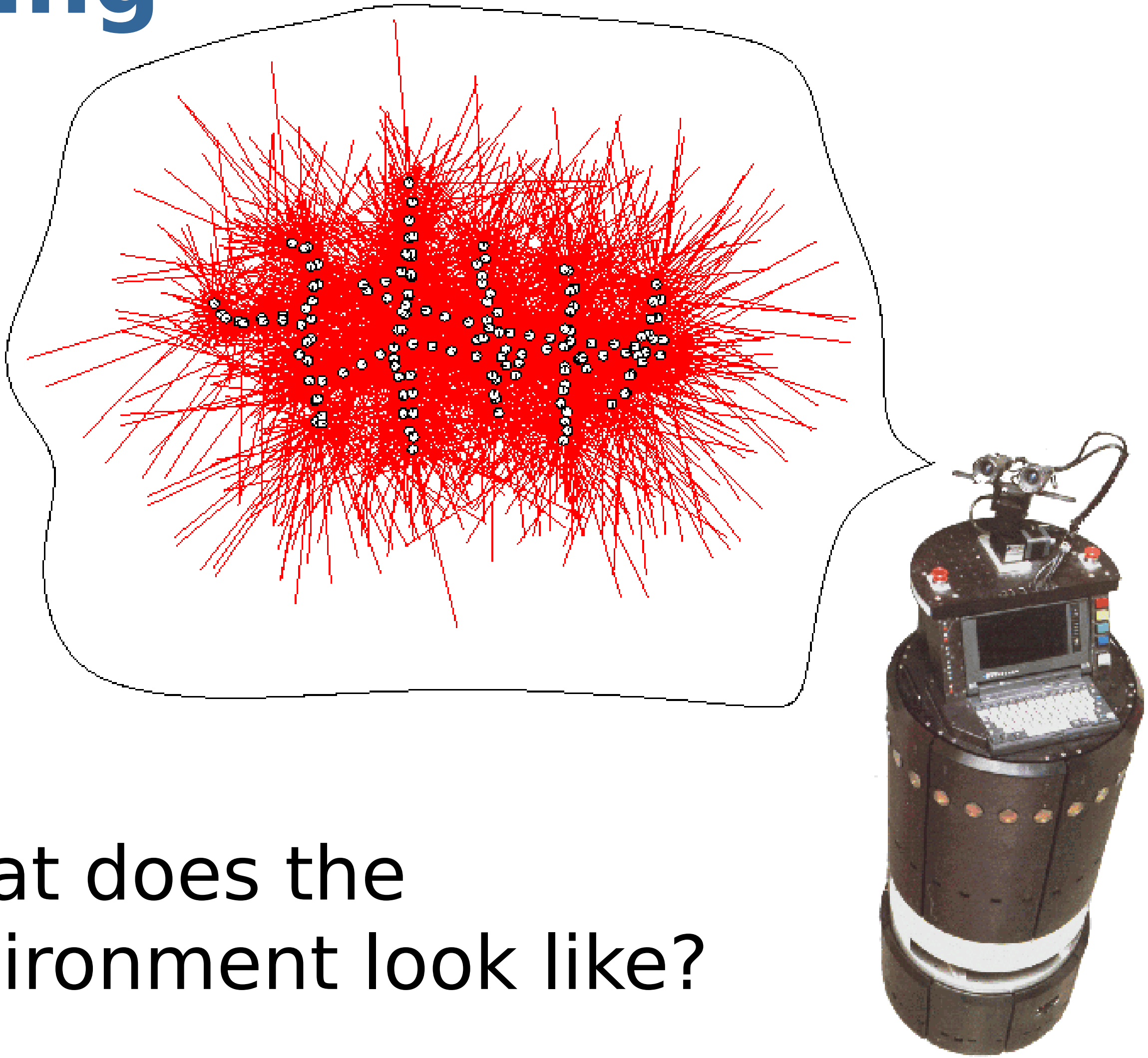
Grid Maps and Mapping With Known Poses

Wolfram Burgard, Cyrill Stachniss, Maren Bennewitz, Diego Tipaldi, Luciano Spinello

Why Mapping?

- Learning maps is one of the fundamental problems in mobile robotics
- Maps allow robots to efficiently carry out their tasks, allow localization …
- Successful robot systems rely on maps for localization, path planning, activity planning etc.

The General Problem of Mapping



What does the environment look like?

The General Problem of Mapping

- Formally, mapping involves, given the sensor data

$$d = \{u_1, z_1, u_2, z_2, \cdots, u_t, z_t\}$$

- to calculate the most likely map

$$m^\star = \mathrm{argmax}_m P(m|d)$$

The General Problem of Mapping

- Formally, mapping involves, given the sensor data

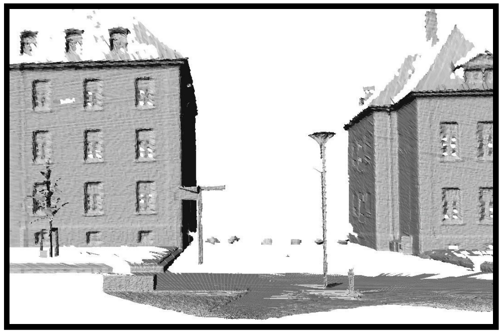
$$d = \{u_1, z_1, u_2, z_2, \cdots, u_t, z_t\}$$

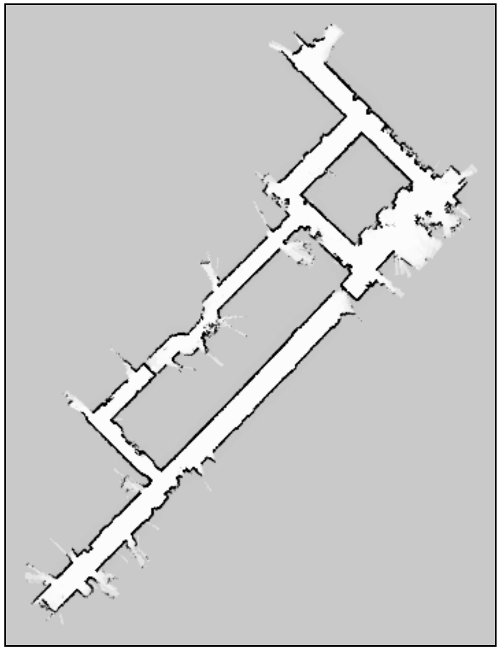
- to calculate the most likely map

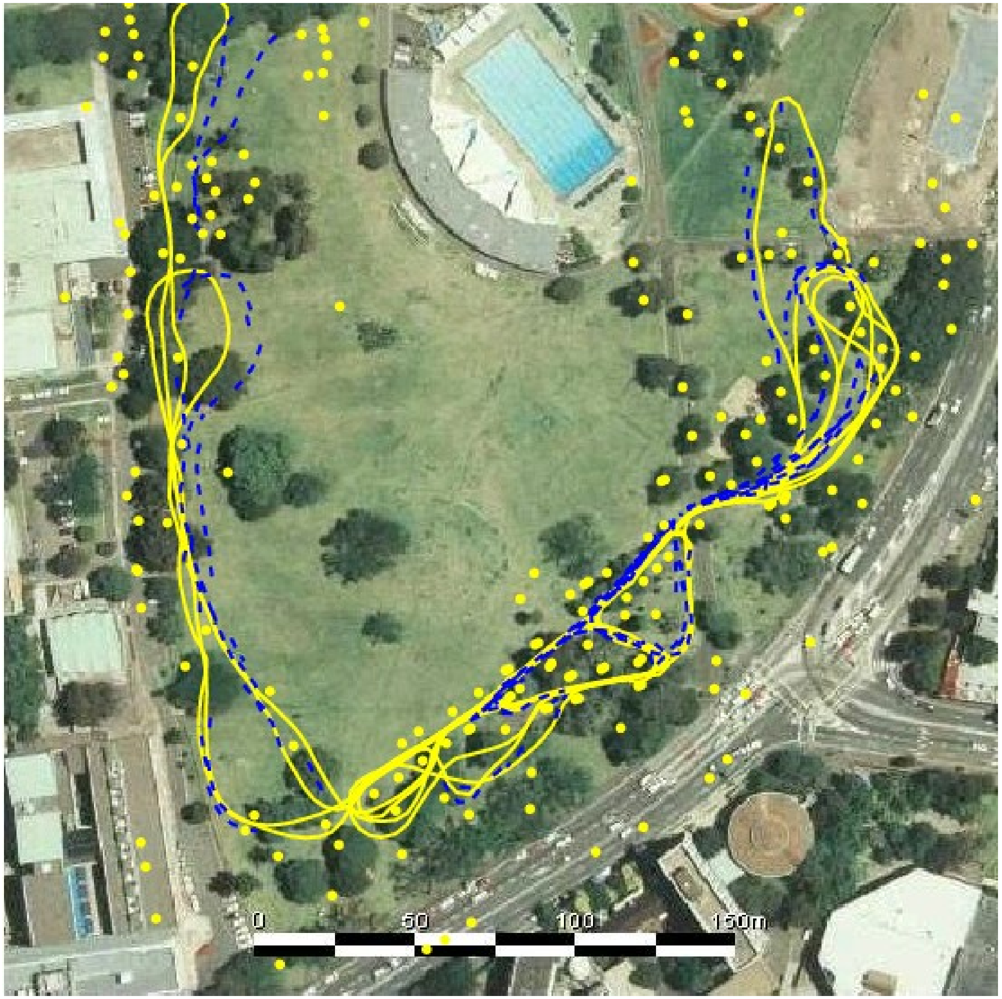
$$m^\star = \mathrm{argmax}_m P(m|d)$$

- Today we describe **how to calculate a map given the robot’s pose**

Features vs. Volumetric Maps





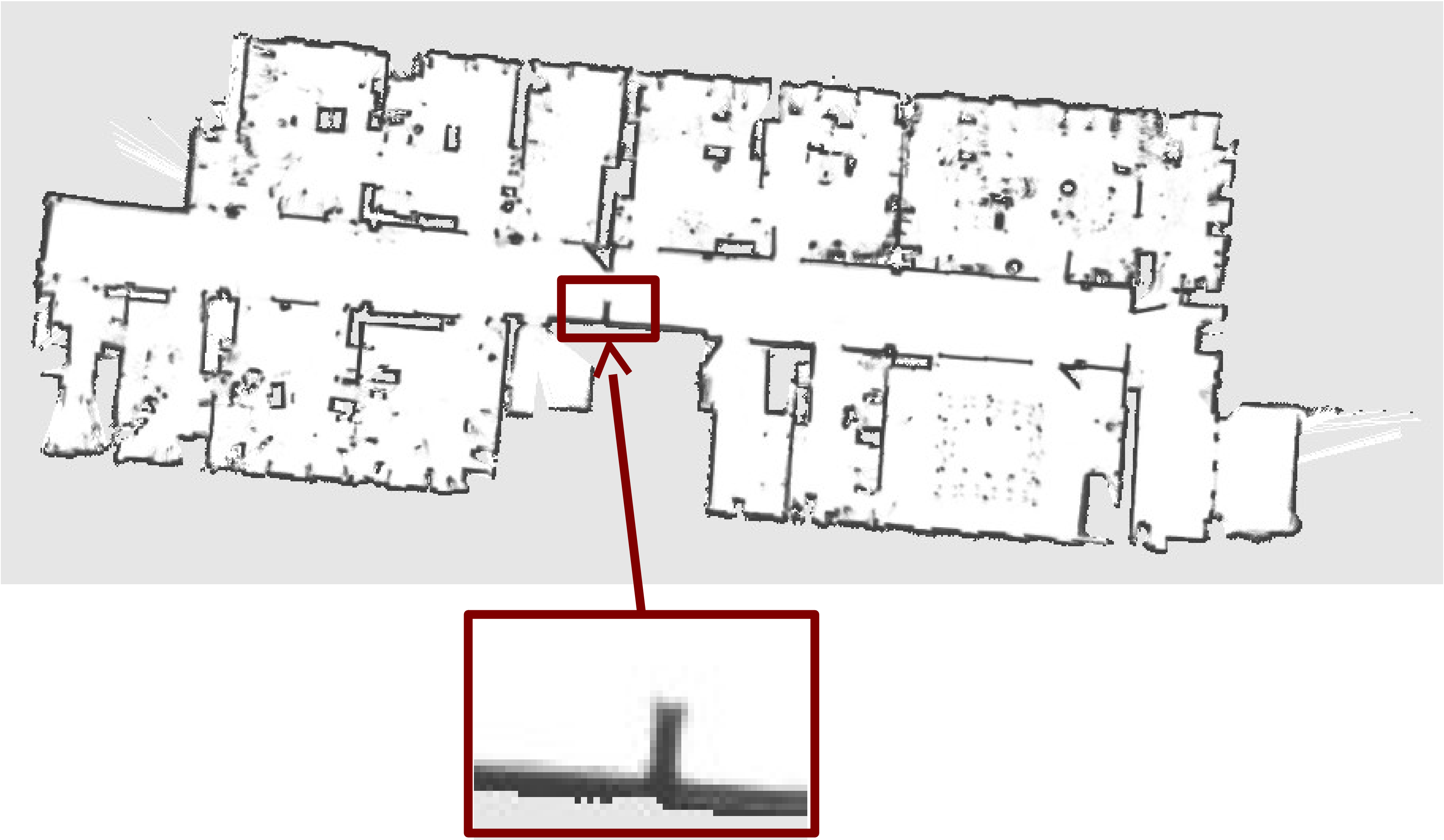


Courtesy by E. Nebot

Grid Maps

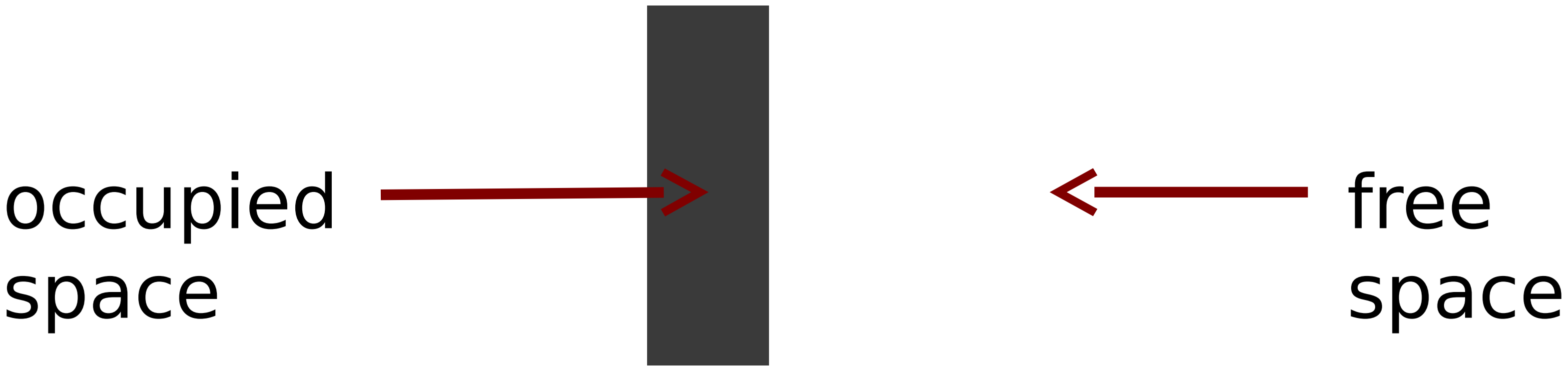
- Discretize the world into cells
- Grid structure is rigid
- Each cell is assumed to be occupied or free space
- Non-parametric model
- Require substantial memory resources
- Does not rely on a feature detector

Example



Assumption 1

- The area that corresponds to a cell is either completely free or occupied

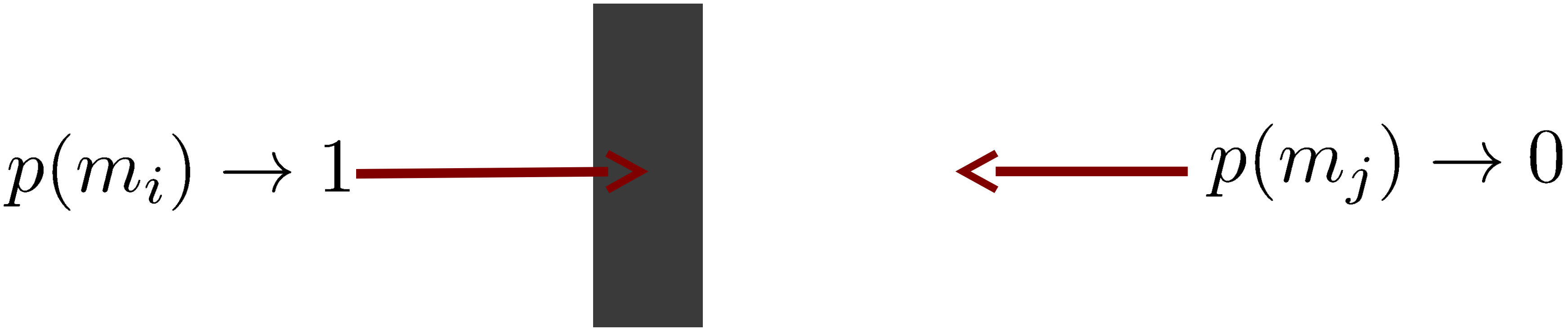


occupied space

free space

Representation

- Each cell is a **binary random variable** that models the occupancy



$p(m_i) \rightarrow 1$

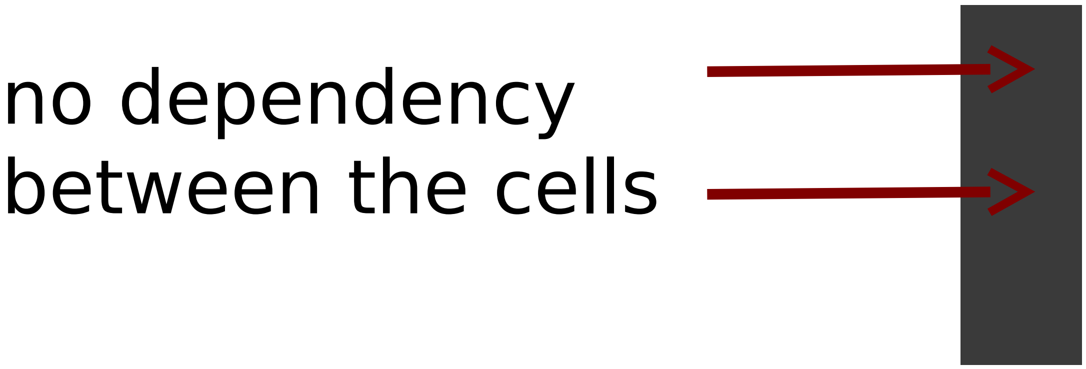
$p(m_j) \rightarrow 0$

Occupancy Probability

- Each cell is a **binary random variable** that models the occupancy
- Cell is occupied $p(m_i) = 1$
- Cell is not occupied $p(m_i) = 0$
- No knowledge $p(m_i) = 0.5$
- The **state** is assumed to be **static**

Assumption 2

- The cells (the random variables) are **independent** of each other



no dependency between the cells

Representation

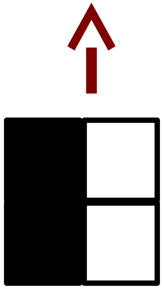
- The probability distribution of the map is given by the product over the cells

$$\underset{\substack{\Big\uparrow \\ \text{map}}}{p(m)} \quad = \quad \prod_i \underset{\substack{\Big\uparrow \\ \text{cell}}}{p(m_i)}$$

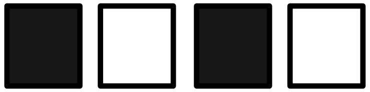
Representation

- The probability distribution of the map is given by the product over the cells

$$p(m) = \prod_i p(m_i)$$



example map
(4-dim vector)



4 individual cells

Estimating a Map From Data

- Given sensor data $z_{1:t}$ and the poses $x_{1:t}$ of the sensor, estimate the map

$$p(m \mid z_{1:t}, x_{1:t}) = \prod_i p(m_i \mid z_{1:t}, x_{1:t})$$

binary random variable

➡ Binary Bayes filter
(for a static state)

Static State Binary Bayes Filter

$$p(m_i \mid z_{1:t}, x_{1:t}) \overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t})\; p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$

Static State Binary Bayes Filter

$$p(m_i \mid z_{1:t}, x_{1:t}) \overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t})\, p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$

$$\overset{\text{Markov}}{=} \frac{p(z_t \mid m_i, x_t)\, p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$

Static State Binary Bayes Filter

$$p(m_i \mid z_{1:t}, x_{1:t}) \overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t})\; p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$

$$\overset{\text{Markov}}{=} \frac{p(z_t \mid m_i, x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})}$$

$$p(z_t \mid m_i, x_t) \overset{\text{Bayes rule}}{=} \frac{p(m_i \mid z_t, x_t)\; p(z_t \mid x_t)}{p(m_i \mid x_t)}$$

Static State Binary Bayes Filter

$$
\begin{aligned}
p(m_i \mid z_{1:t}, x_{1:t}) &\overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t})\; p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Markov}}{=} \frac{p(z_t \mid m_i, x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Bayes rule}}{=} \frac{p(m_i \mid z_t, x_t)\; p(z_t \mid x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i \mid x_t)\; p(z_t \mid z_{1:t-1}, x_{1:t})}
\end{aligned}
$$

Static State Binary Bayes Filter

$$
\begin{aligned}
p(m_i \mid z_{1:t}, x_{1:t}) &\overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t}) \, p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Markov}}{=} \frac{p(z_t \mid m_i, x_t) \, p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Bayes rule}}{=} \frac{p(m_i \mid z_t, x_t) \, p(z_t \mid x_t) \, p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i \mid x_t) \, p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Markov}}{=} \frac{p(m_i \mid z_t, x_t) \, p(z_t \mid x_t) \, p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i) \, p(z_t \mid z_{1:t-1}, x_{1:t})}
\end{aligned}
$$

Static State Binary Bayes Filter

$$
\begin{aligned}
p(m_i \mid z_{1:t}, x_{1:t}) &\overset{\text{Bayes rule}}{=} \frac{p(z_t \mid m_i, z_{1:t-1}, x_{1:t})\; p(m_i \mid z_{1:t-1}, x_{1:t})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Markov}}{=} \frac{p(z_t \mid m_i, x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Bayes rule}}{=} \frac{p(m_i \mid z_t, x_t)\; p(z_t \mid x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i \mid x_t)\; p(z_t \mid z_{1:t-1}, x_{1:t})} \\
&\overset{\text{Markov}}{=} \frac{p(m_i \mid z_t, x_t)\; p(z_t \mid x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})}{p(m_i)\; p(z_t \mid z_{1:t-1}, x_{1:t})}
\end{aligned}
$$

Do exactly the same for the opposite event:

$$
p(\neg m_i \mid z_{1:t}, x_{1:t}) \overset{\text{the same}}{=} \frac{p(\neg m_i \mid z_t, x_t)\; p(z_t \mid x_t)\; p(\neg m_i \mid z_{1:t-1}, x_{1:t-1})}{p(\neg m_i)\; p(z_t \mid z_{1:t-1}, x_{1:t})}
$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{p(\neg m_i \mid z_{1:t}, x_{1:t})} = \frac{\dfrac{p(m_i|z_t,x_t)\;\cancel{p(z_t|x_t)}\;p(m_i|z_{1:t-1},x_{1:t-1})}{p(m_i)\;\cancel{p(z_t|z_{1:t-1},x_{1:t})}}}{\dfrac{p(\neg m_i|z_t,x_t)\;\cancel{p(z_t|x_t)}\;p(\neg m_i|z_{1:t-1},x_{1:t-1})}{p(\neg m_i)\;\cancel{p(z_t|z_{1:t-1},x_{1:t})}}}$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{p(\neg m_i \mid z_{1:t}, x_{1:t})}$$

$$= \frac{p(m_i \mid z_t, x_t)\; p(m_i \mid z_{1:t-1}, x_{1:t-1})\; p(\neg m_i)}{p(\neg m_i \mid z_t, x_t)\; p(\neg m_i \mid z_{1:t-1}, x_{1:t-1})\; p(m_i)}$$

$$= \frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)} \frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})} \frac{1 - p(m_i)}{p(m_i)}$$

Static State Binary Bayes Filter

- By computing the ratio of both probabilities, we obtain:

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{1 - p(m_i \mid z_{1:t}, x_{1:t})}$$

$$= \frac{p(m_i \mid z_t, x_t)\, p(m_i \mid z_{1:t-1}, x_{1:t-1})\, p(\neg m_i)}{p(\neg m_i \mid z_t, x_t)\, p(\neg m_i \mid z_{1:t-1}, x_{1:t-1})\, p(m_i)}$$

$$= \underbrace{\frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)}}_{\text{uses } z_t} \underbrace{\frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})}}_{\text{recursive term}} \underbrace{\frac{1 - p(m_i)}{p(m_i)}}_{\text{prior}}$$

Occupancy Update Rule

- Recursive rule

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{1 - p(m_i \mid z_{1:t}, x_{1:t})}$$

$$= \underbrace{\frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)}}_{\text{uses } z_t} \underbrace{\frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})}}_{\text{recursive term}} \underbrace{\frac{1 - p(m_i)}{p(m_i)}}_{\text{prior}}$$

Occupancy Update Rule

- Recursive rule

$$\frac{p(m_i \mid z_{1:t}, x_{1:t})}{1 - p(m_i \mid z_{1:t}, x_{1:t})} = \underbrace{\frac{p(m_i \mid z_t, x_t)}{1 - p(m_i \mid z_t, x_t)}}_{\text{uses } z_t} \underbrace{\frac{p(m_i \mid z_{1:t-1}, x_{1:t-1})}{1 - p(m_i \mid z_{1:t-1}, x_{1:t-1})}}_{\text{recursive term}} \underbrace{\frac{1 - p(m_i)}{p(m_i)}}_{\text{prior}}$$

- Often written as

$$Bel(m_t^i) = \left[1 + \frac{1 - p(m_t^i|z_t, u_{t-1})}{p(m_t^i|z_t, u_{t-1})} \frac{p(m_t^i)}{1 - p(m_t^i)} \frac{1 - Bel(m_{t-1}^i)}{Bel(m_{t-1}^i)}\right]^{-1}$$

Log Odds Notation

- Log odds ratio is defined as

$$l(x) \quad = \quad \log \frac{p(x)}{1 - p(x)}$$

- and with the ability to retrieve $p(x)$

$$p(x) \quad = \quad 1 - \frac{1}{1 + \exp l(x)}$$

Occupancy Mapping in Log Odds Form

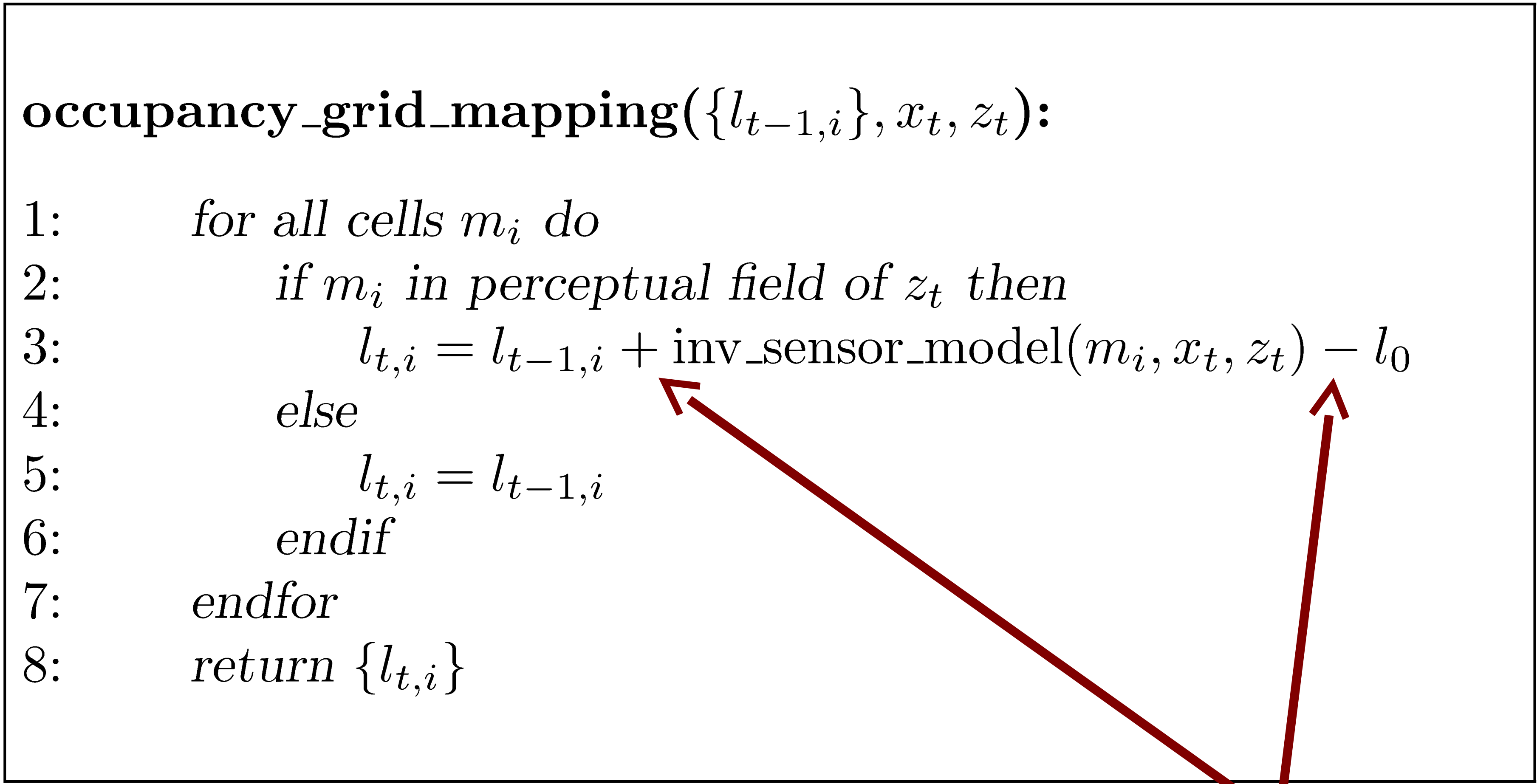
- The product turns into a sum

$$l(m_i \mid z_{1:t}, x_{1:t}) = \underbrace{l(m_i \mid z_t, x_t)}_{\text{inverse sensor model}} + \underbrace{l(m_i \mid z_{1:t-1}, x_{1:t-1})}_{\text{recursive term}} - \underbrace{l(m_i)}_{\text{prior}}$$

- or in short

$$l_{t,i} = \text{inv\_sensor\_model}(m_i, x_t, z_t) + l_{t-1,i} - l_0$$

Occupancy Mapping Algorithm



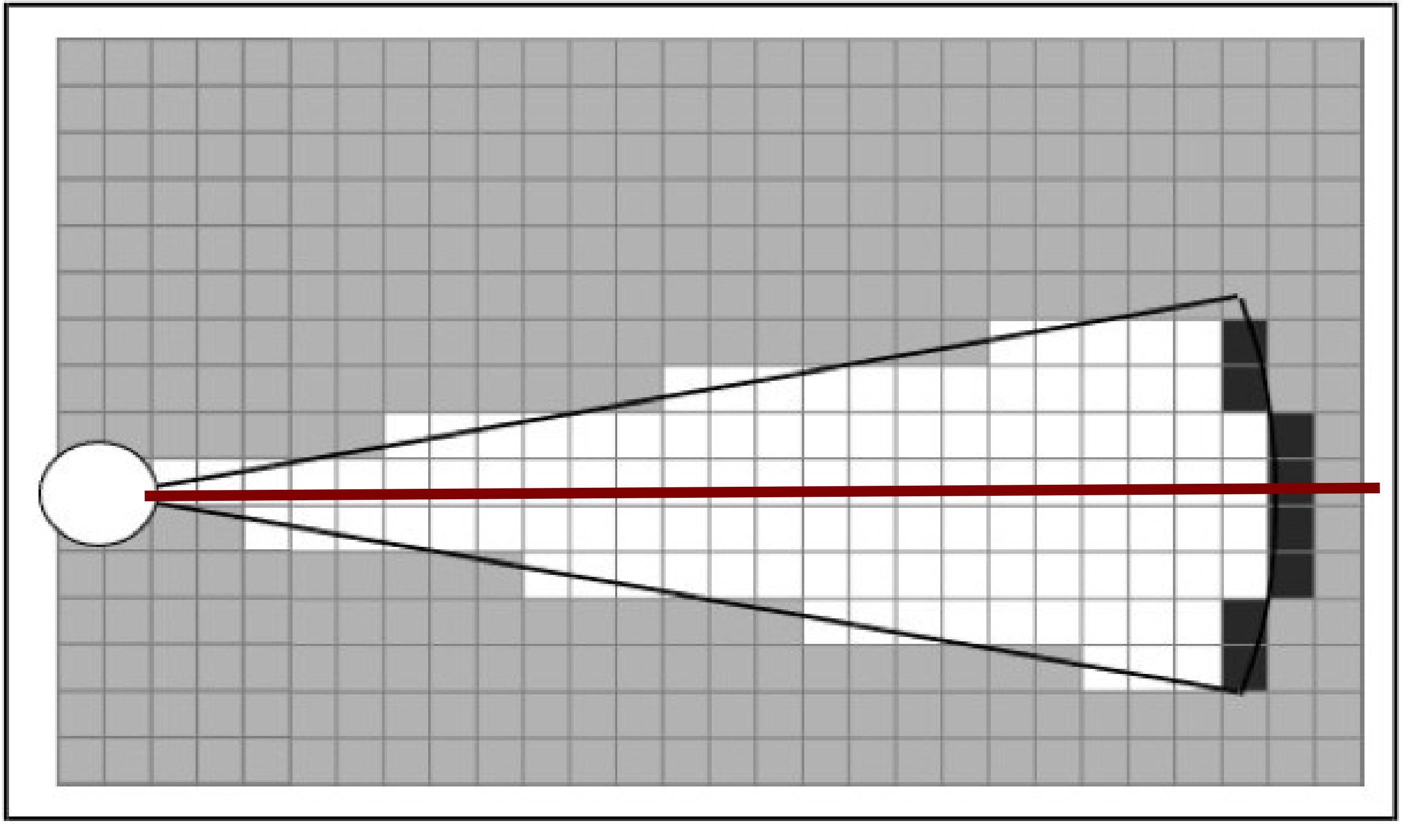
```
occupancy_grid_mapping($\{l_{t-1,i}\}, x_t, z_t$):

for all cells $m_i$ do
    if $m_i$ in perceptual field of $z_t$ then
        $l_{t,i} = l_{t-1,i} + \text{inv\_sensor\_model}(m_i, x_t, z_t) - l_0$
    else
        $l_{t,i} = l_{t-1,i}$
    endif
endfor
return $\{l_{t,i}\}$
```

highly efficient, only requires to compute sums

Occupancy Grid Mapping

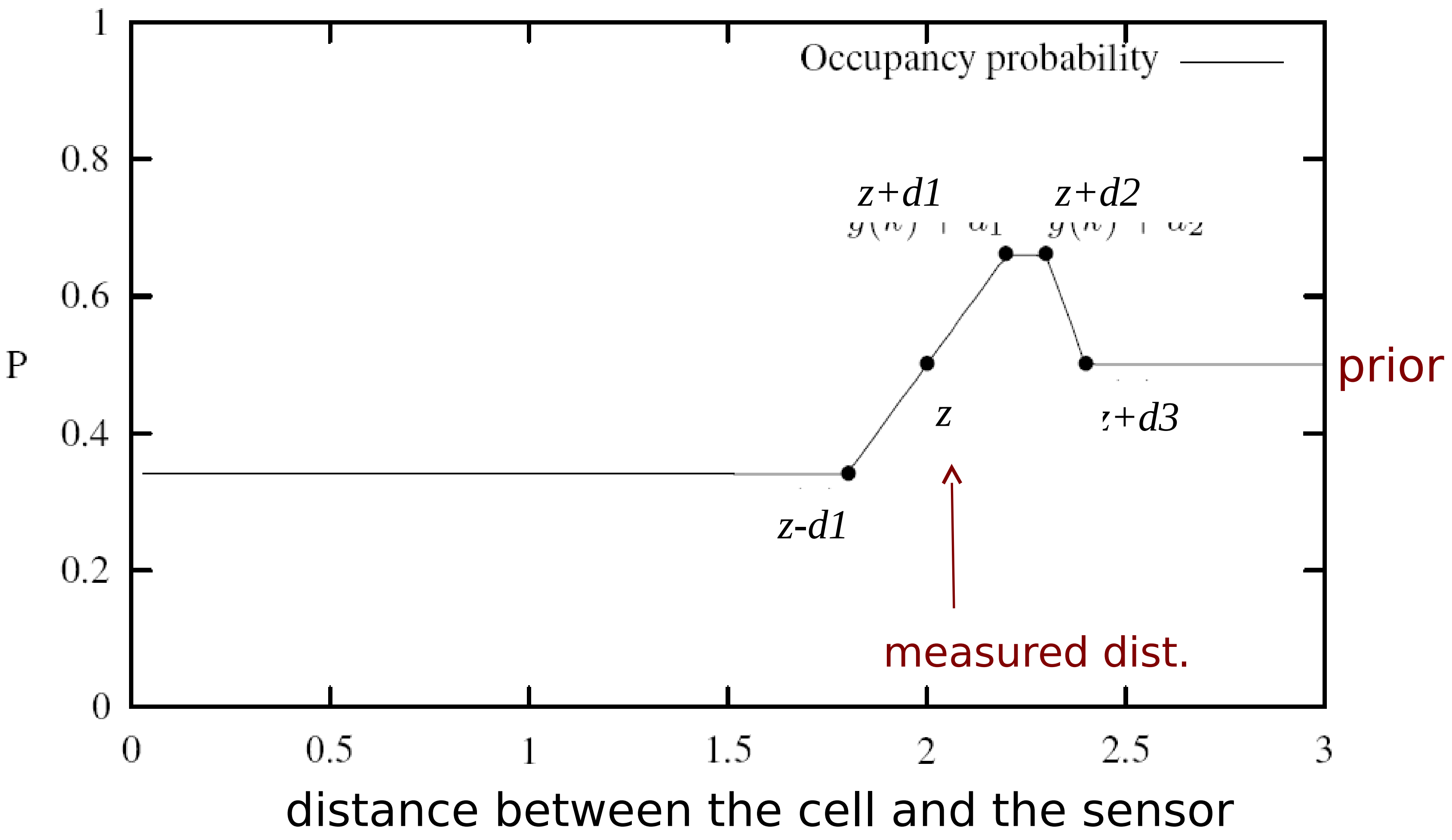
- Developed in the mid 80’s by Moravec and Elfes
- Originally developed for noisy sonars
- Also called “mapping with know poses”

Inverse Sensor Model for Sonars Range Sensors



In the following, consider the cells along the optical axis (red line)

Occupancy Value Depending on the Measured Distance

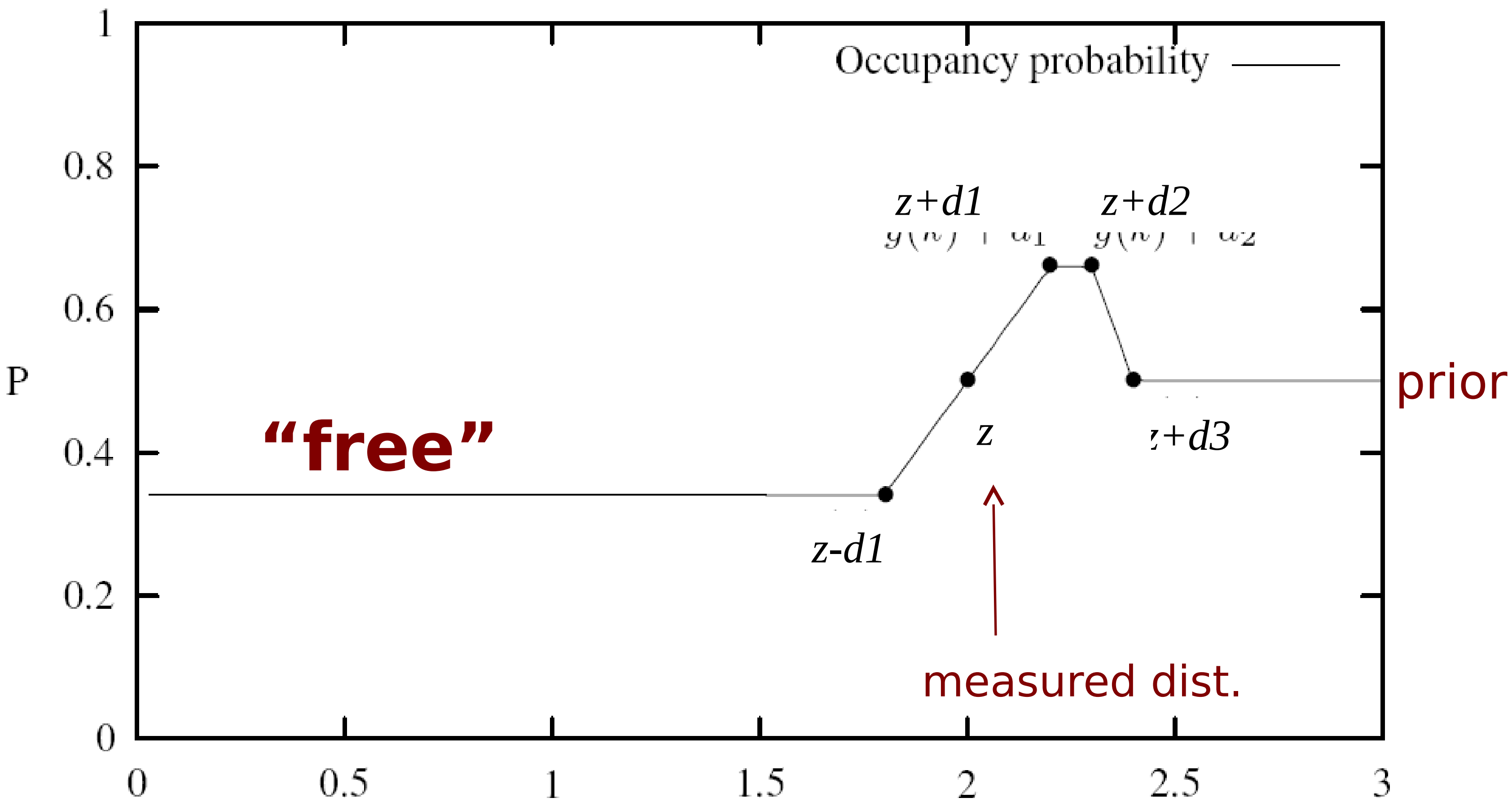


prior

measured dist.

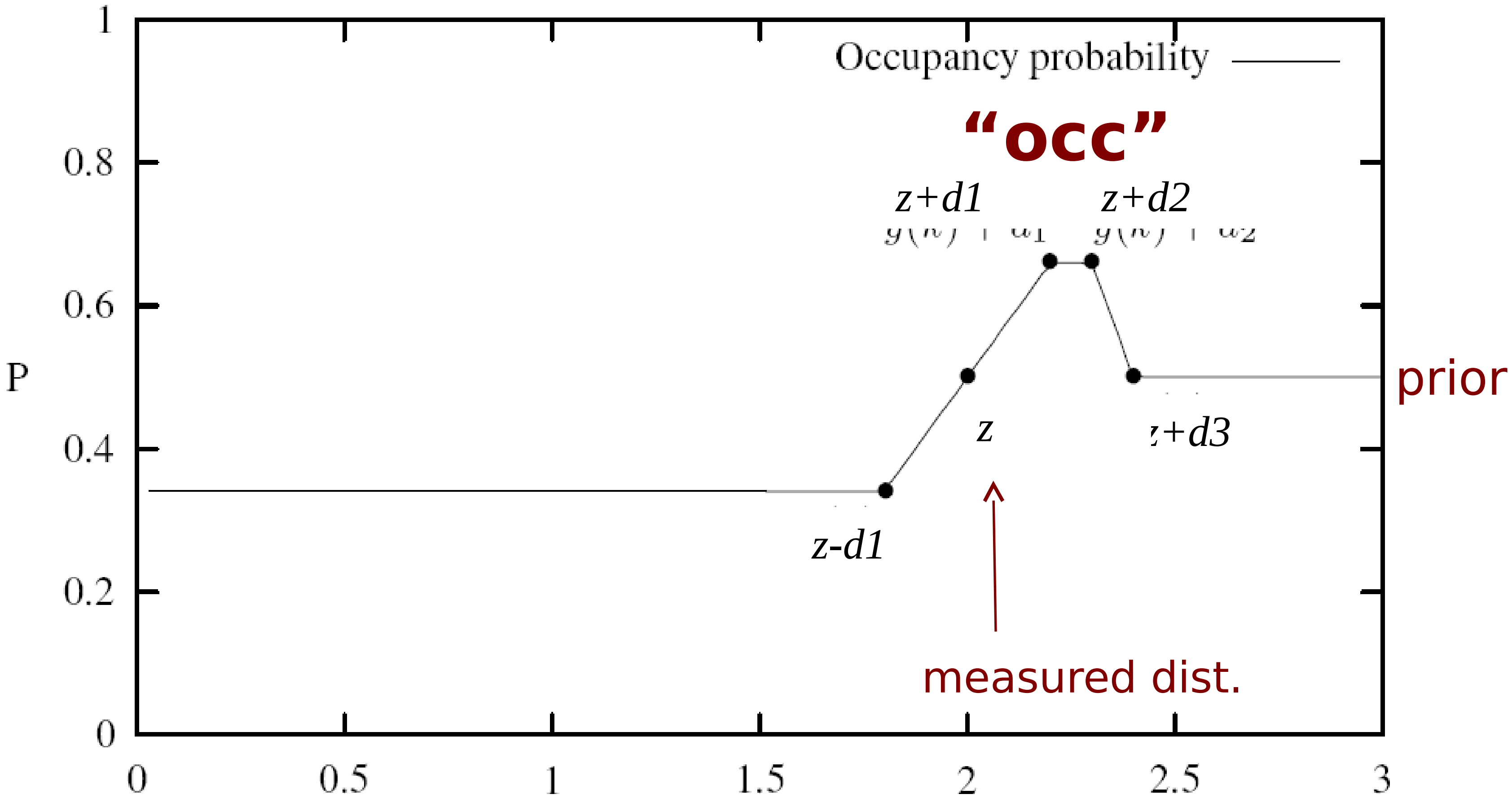
distance between the cell and the sensor

Occupancy Value Depending on the Measured Distance



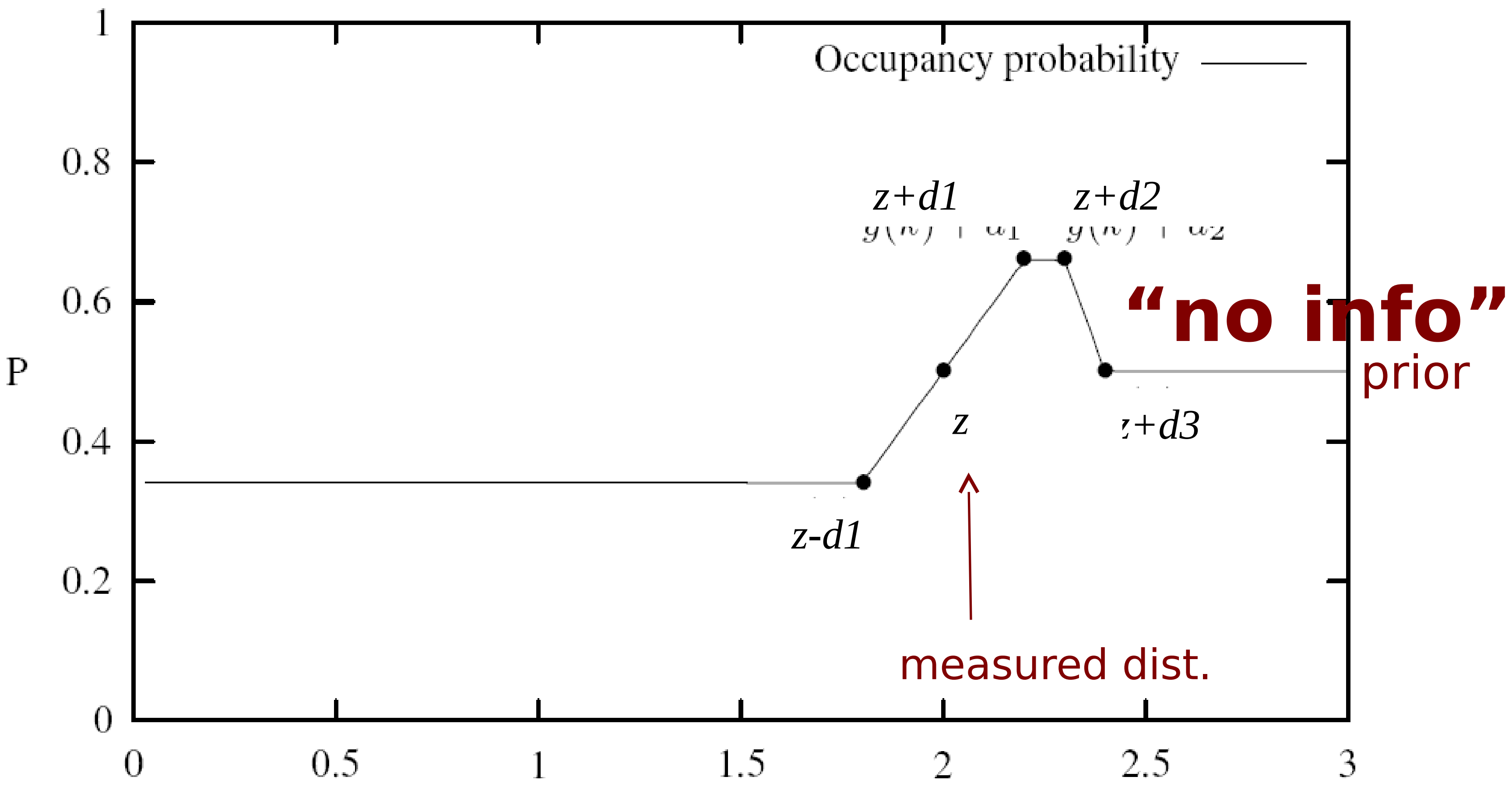
distance between the cell and the sensor

Occupancy Value Depending on the Measured Distance



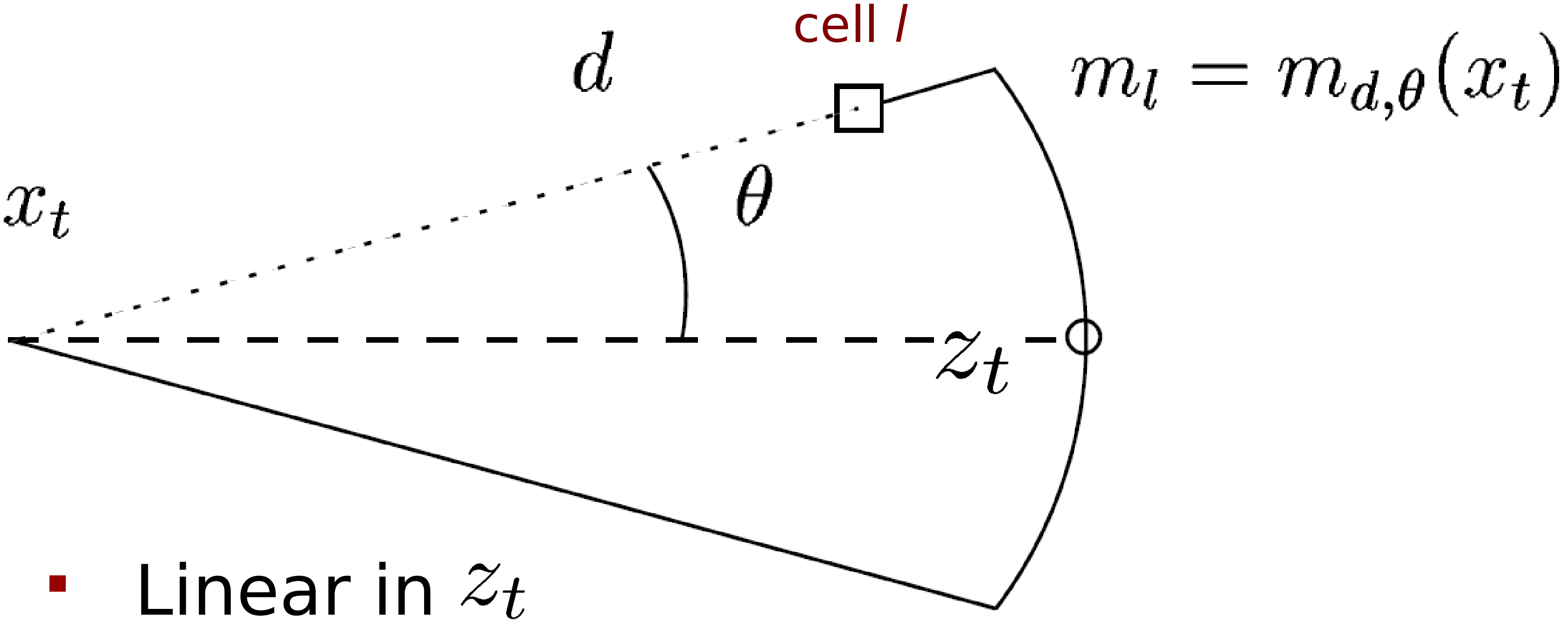
distance between the cell and the sensor

Occupancy Value Depending on the Measured Distance



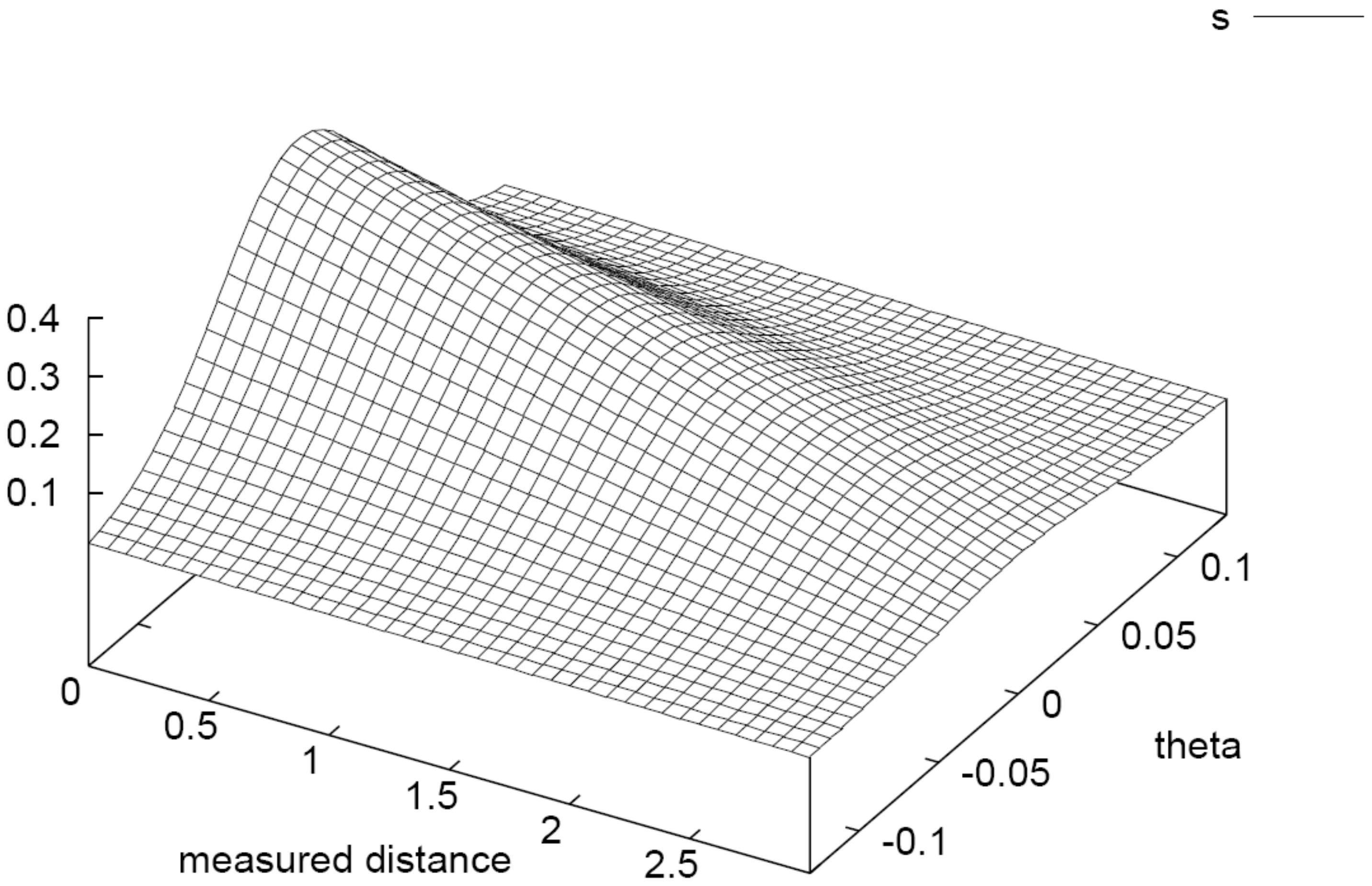
distance between the cell and the sensor

Update depends on the Measured Distance and Deviation from the Optical Axis

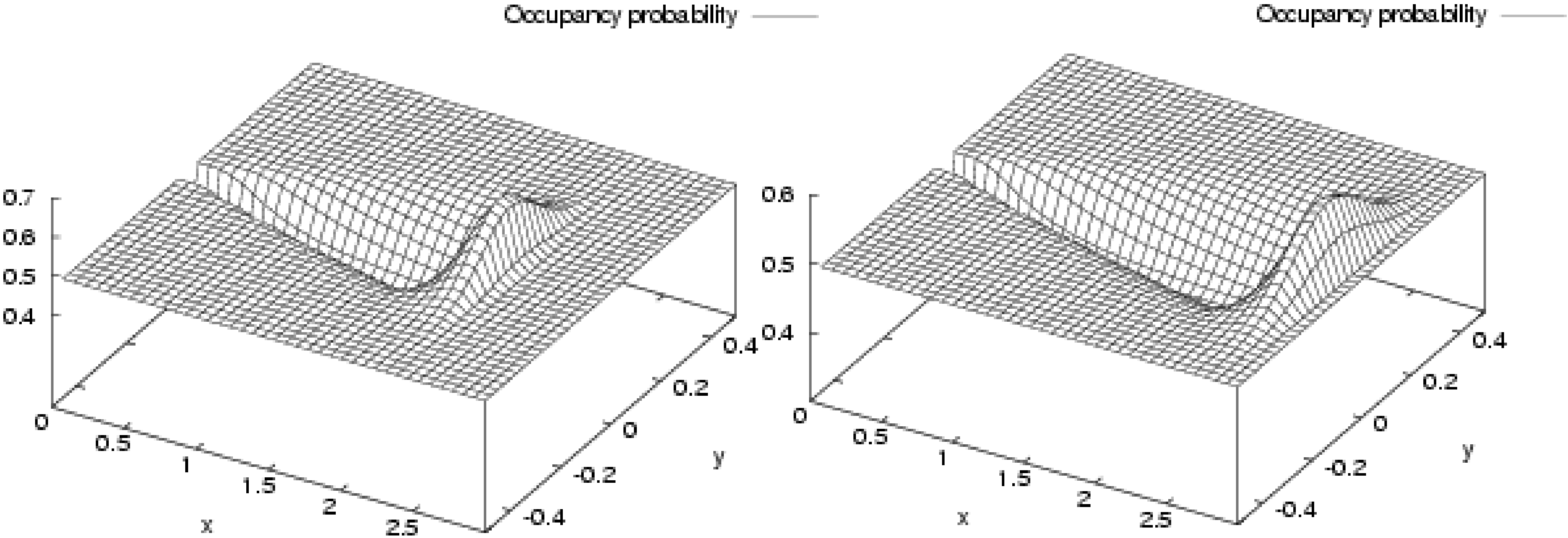


- Linear in z_t
- Gaussian in θ

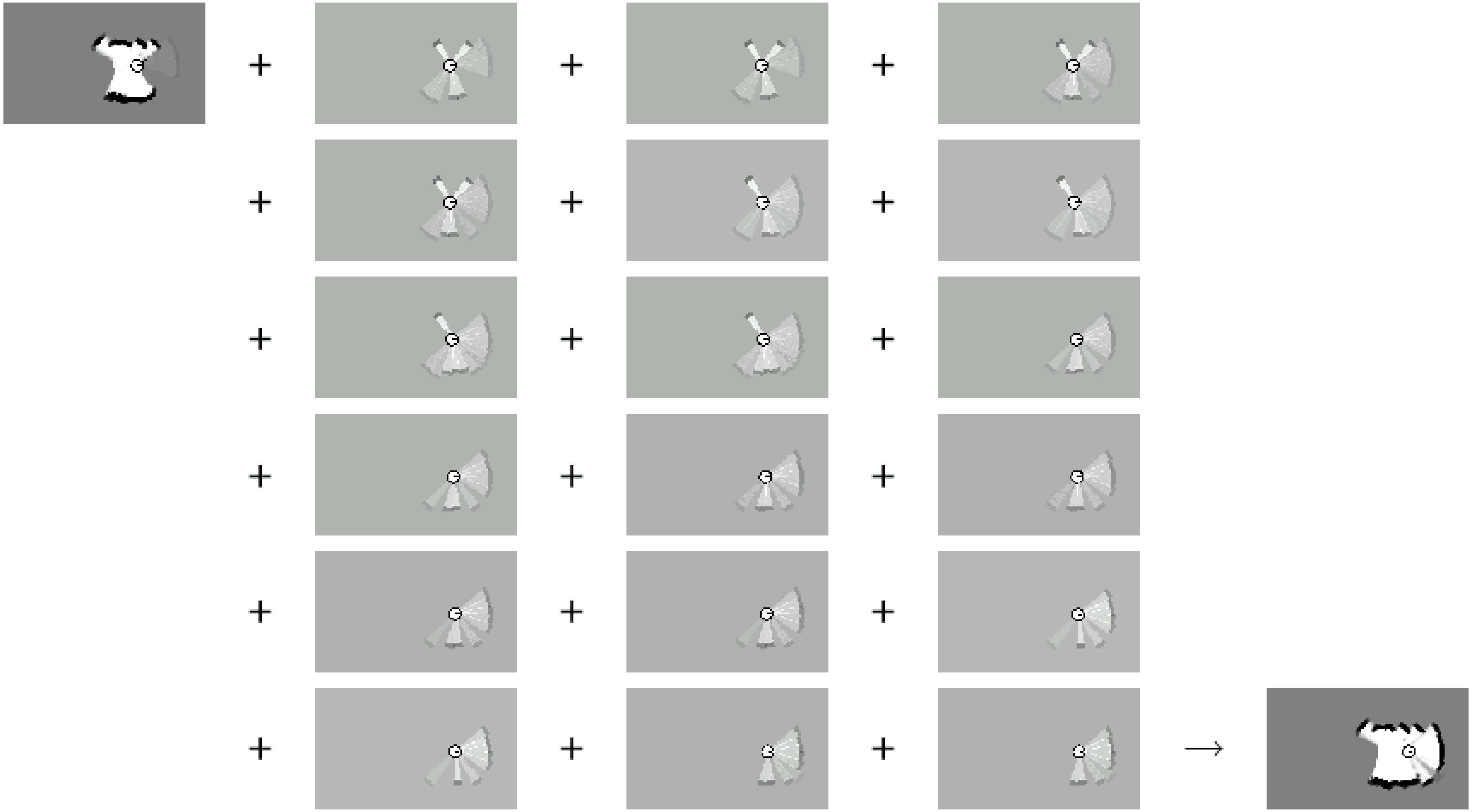
Intensity of the Update



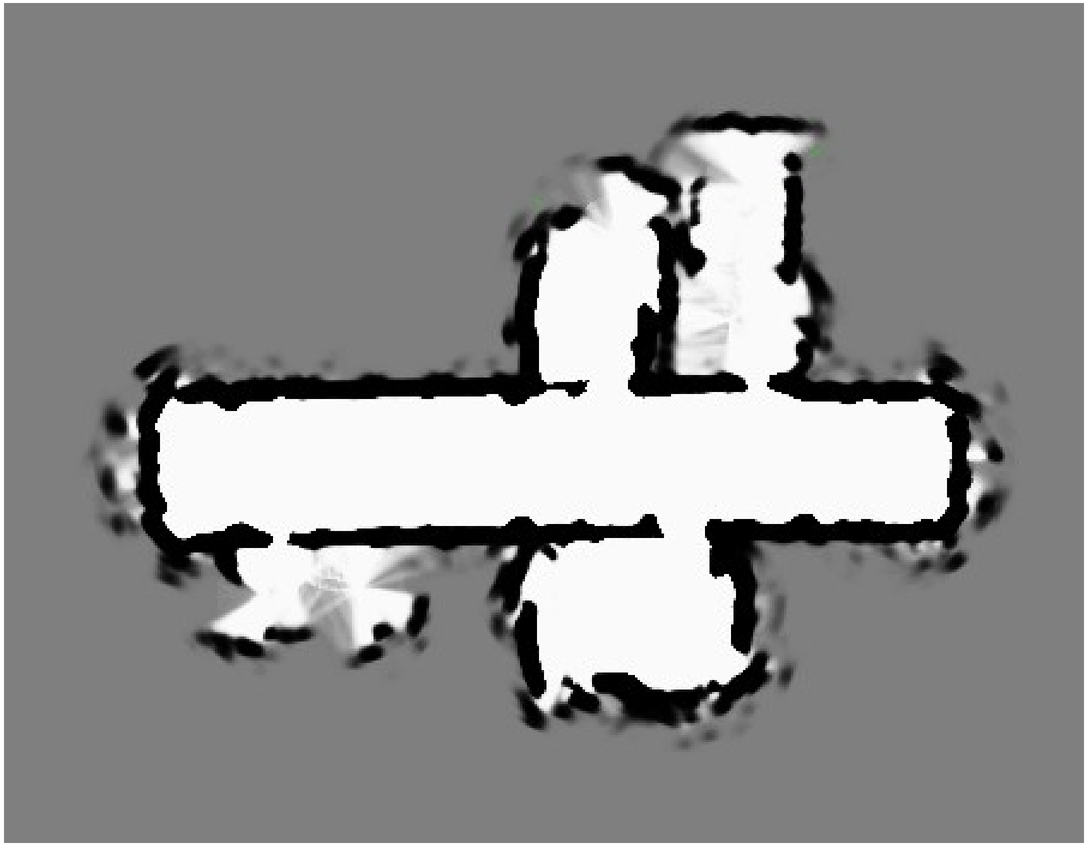
Resulting Model



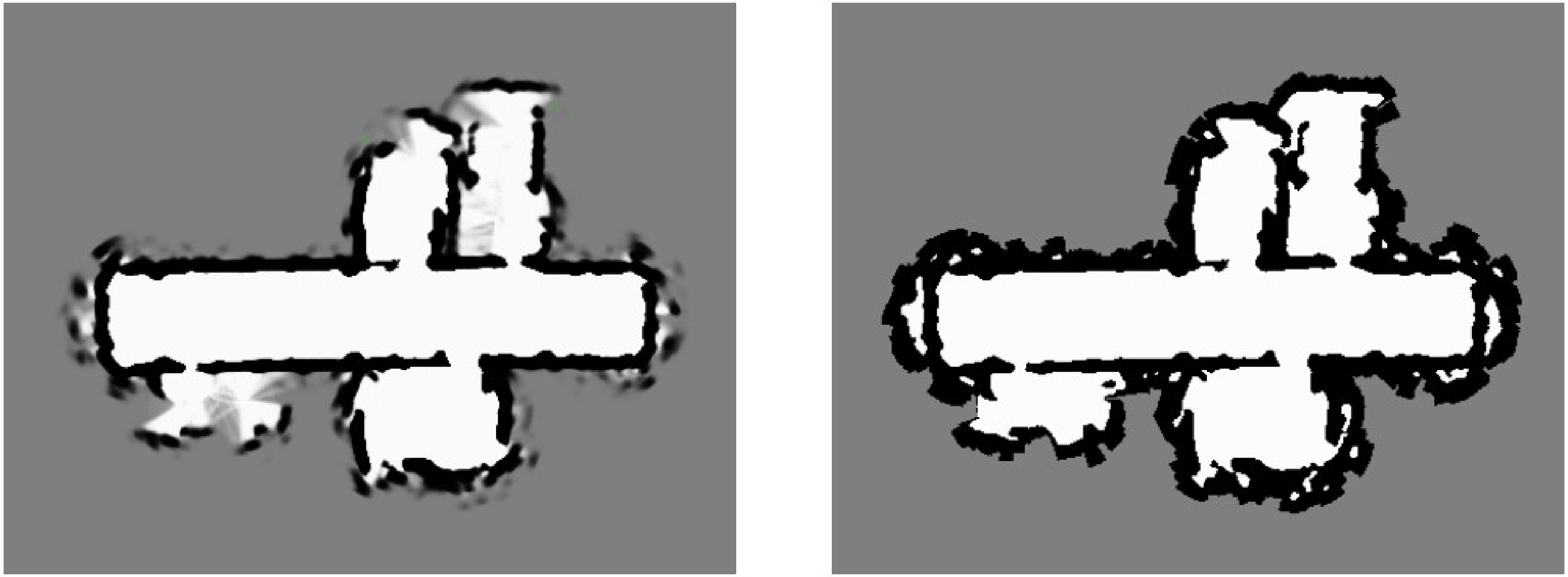
Example: Incremental Updating of Occupancy Grids



Resulting Map Obtained with Ultrasound Sensors

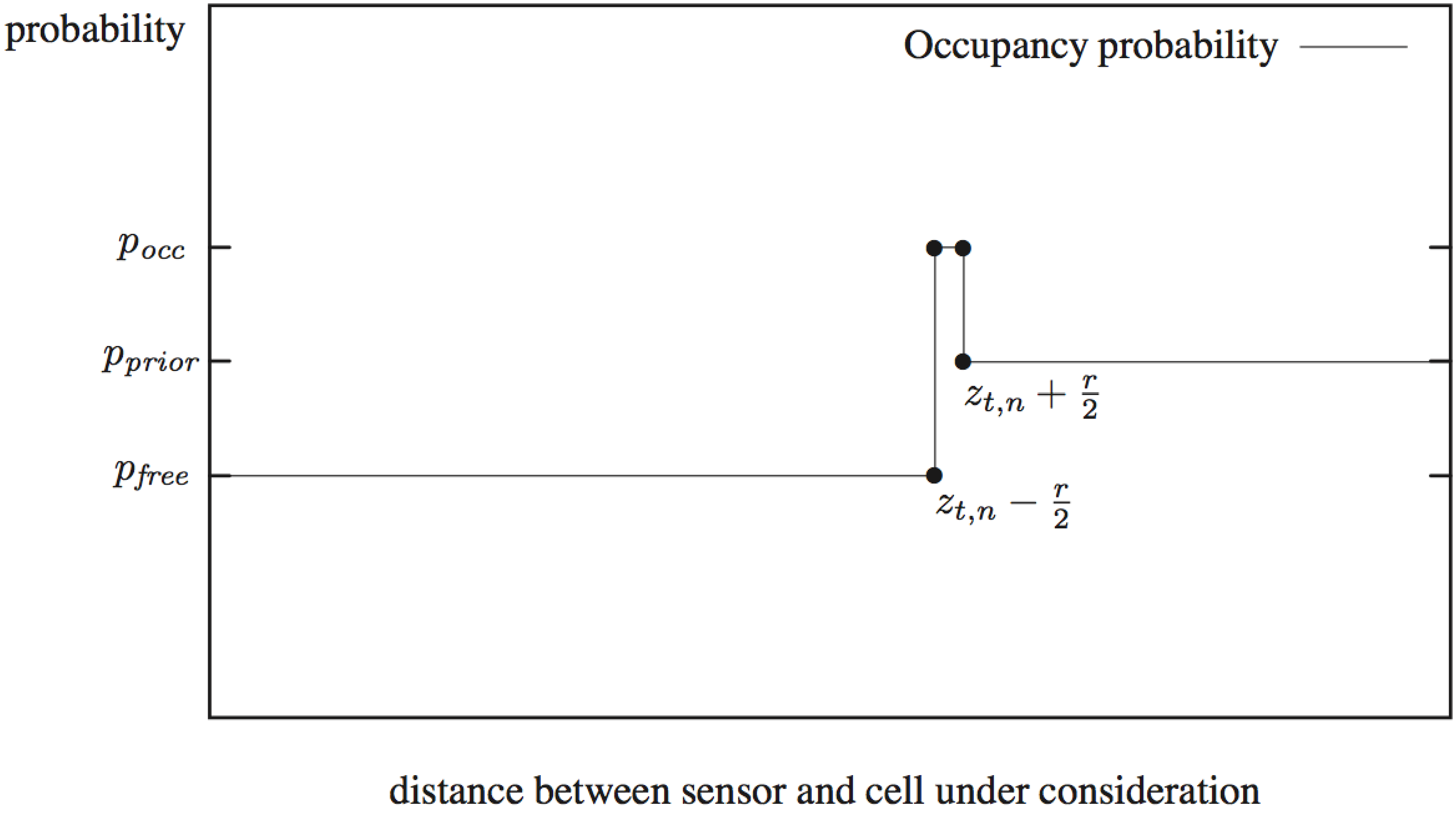


Resulting Occupancy and Maximum Likelihood Map

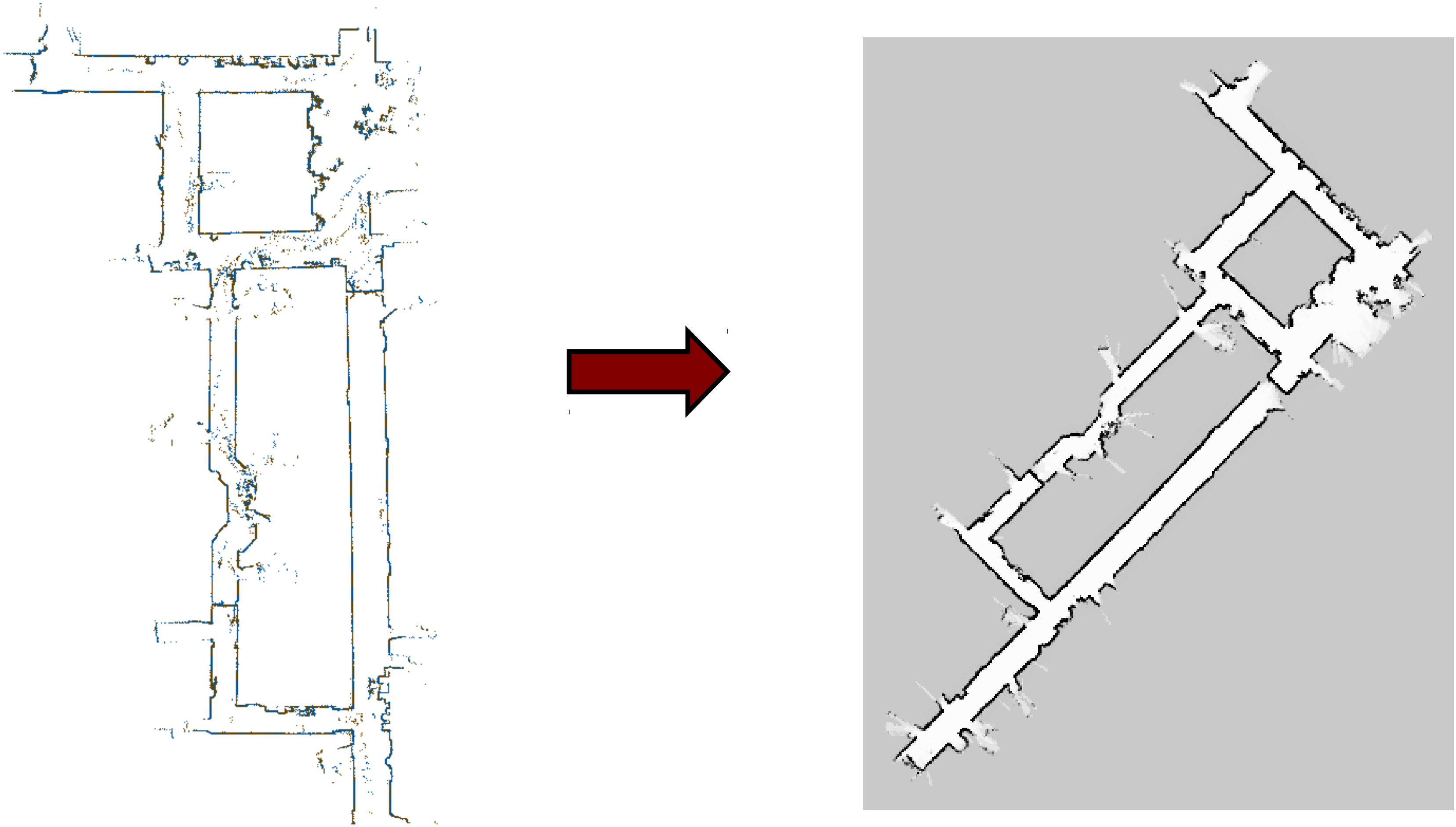


The maximum likelihood map is obtained by rounding the probability for each cell to 0 or 1.

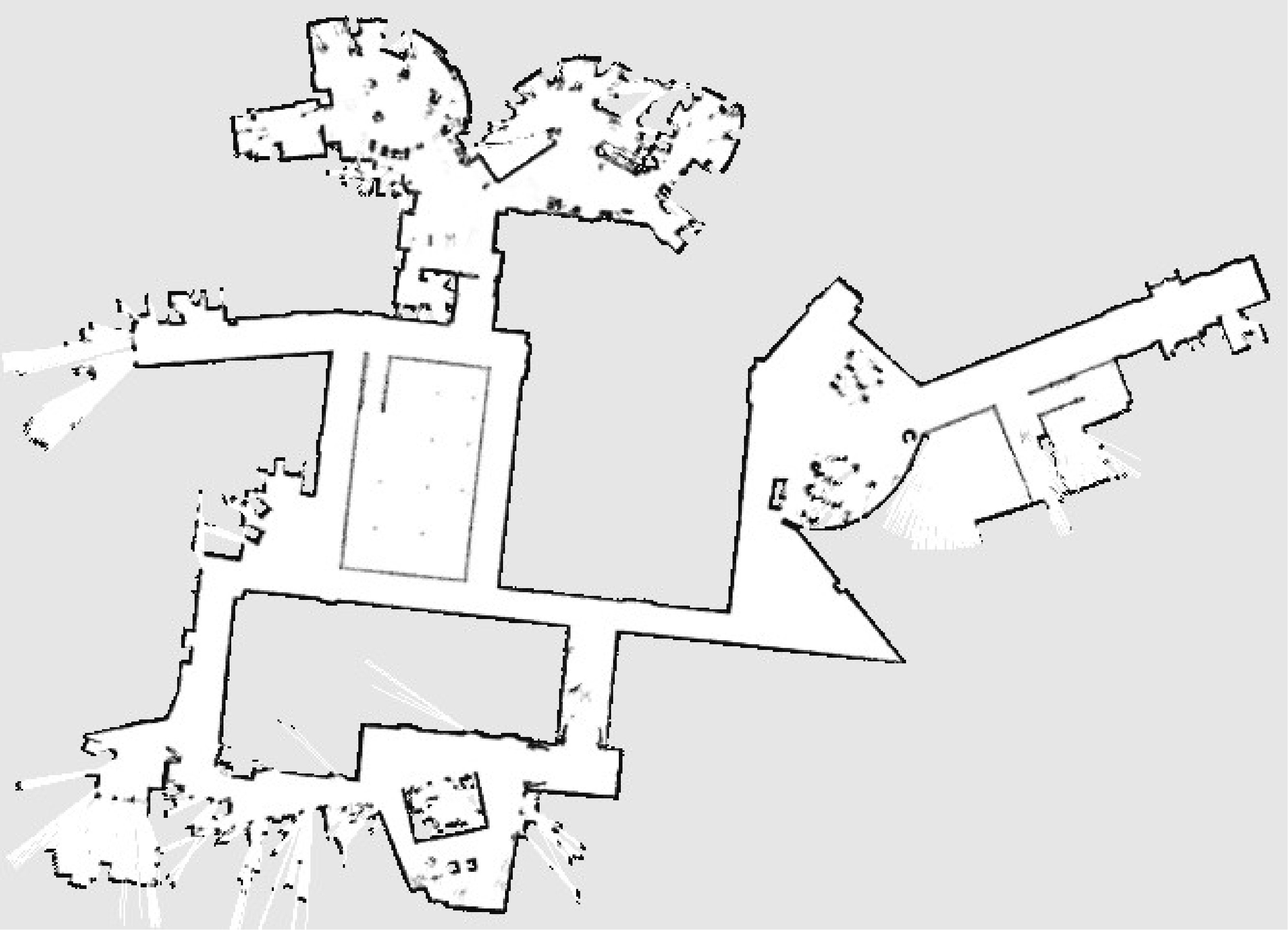
Inverse Sensor Model for Laser Range Finders



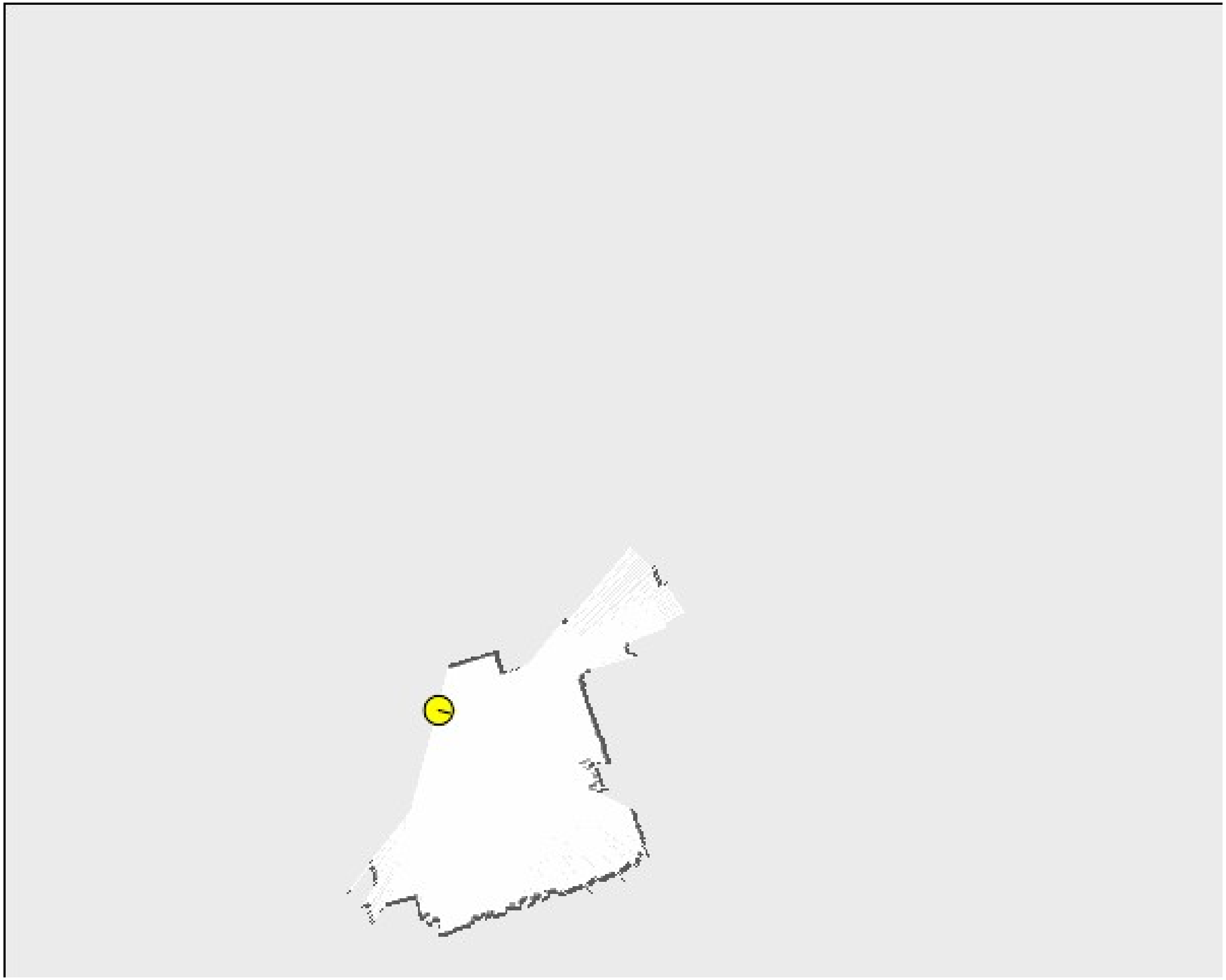
Occupancy Grids
From Laser Scans to Maps



Example: MIT CSAIL 3rd Floor



Uni Freiburg Building 106



Alternative: Counting Model

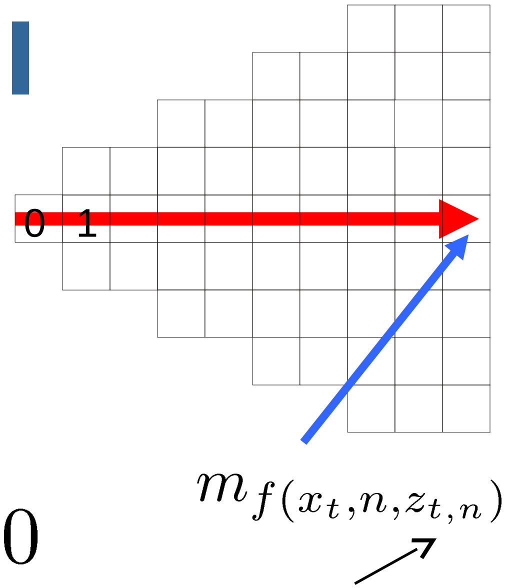
- For every cell count
  - hits(x,y): number of cases where a beam ended at <x,y>
  - misses(x,y): number of cases where a beam passed through <x,y>

$$Bel(m^{[xy]}) = \frac{\text{hits}(x,y)}{\text{hits}(x,y)+\text{misses}(x,y)}$$

- Value of interest: P(reflects(x,y))

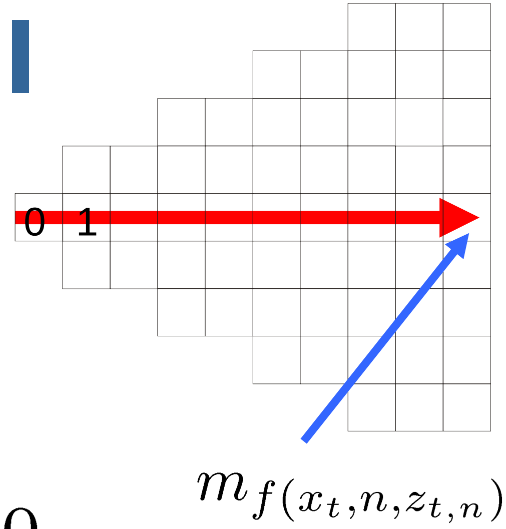
The Measurement Model

- Pose at time *t*: x_t
- Beam *n* of scan at time *t*: $z_{t,n}$
- Maximum range reading: $\zeta_{t,n} = 1$
- Beam reflected by an object: $\zeta_{t,n} = 0$



measured dist. in #cells

The Measurement Model

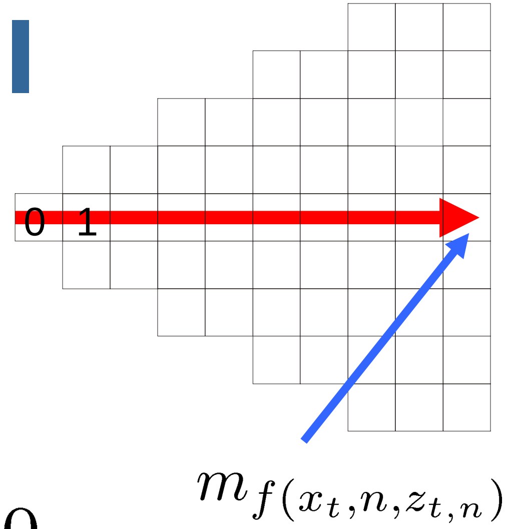


- Pose at time *t*: x_t
- Beam *n* of scan at time *t*: $z_{t,n}$
- Maximum range reading: $\zeta_{t,n} = 1$
- Beam reflected by an object: $\zeta_{t,n} = 0$

max range: “first zt,n-1 cells covered by the beam must be free”

$$p(z_{t,n} | x_t, m) = \begin{cases} \prod_{k=0}^{z_{t,n}-1} (1 - m_{f(x_t,n,k)}) & \text{if } \zeta_{t,n} = 1 \\ & \end{cases}$$

The Measurement Model



- Pose at time *t*: x_t
- Beam *n* of scan at time *t*: $z_{t,n}$
- Maximum range reading: $\zeta_{t,n} = 1$
- Beam reflected by an object: $\zeta_{t,n} = 0$

max range: “first zt,n-1 cells covered by the beam must be free”

$$p(z_{t,n}|x_t,m) = \begin{cases} \prod_{k=0}^{z_{t,n}-1} (1 - m_{f(x_t,n,k)}) & \text{if } \zeta_{t,n} = 1 \\ m_{f(x_t,n,z_{t,n})} \prod_{k=0}^{z_{t,n}-1} (1 - m_{f(x_t,n,k)}) & \zeta_{t,n} = 0 \end{cases}$$

otherwise: “last cell reflected beam, all others free”

Computing the Most Likely Map

- Compute values for m that maximize

$$m^\star = \mathrm{argmax}_m P(m|z_1, \cdots, z_t, x_1, \cdots, x_t)$$

- Assuming a uniform prior probability for P(m), this is equivalent to maximizing:

$$\begin{aligned} m^\star &= \mathrm{argmax}_m P(z_1, \cdots, z_t | m, x_1, \cdots, x_t) \\ &= \mathrm{argmax}_m \prod_{t=1}^{T} P(z_t | m, x_t) \\ &= \mathrm{argmax}_m \sum_{t=1}^{T} \ln P(z_t | m, x_t) \end{aligned}$$

since z_t independent and only depend on x_t

Computing the Most Likely Map

$$
\begin{aligned}
m^\star \quad = \quad & \operatorname{argmax}_m \overset{\text{cells}}{\sum_{j=1}^{J}} \sum_{t=1}^{T} \overset{\text{beams}}{\sum_{n=1}^{N}} \Big(I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n}) \cdot \ln m_j \\
& + \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j) \cdot \ln(1 - m_j) \Big)
\end{aligned}
$$

Computing the Most Likely Map

"beam *n* ends in cell *j*"

$$
\begin{aligned}
m^\star \quad = \quad & \operatorname{argmax}_m \sum_{j=1}^{J} \sum_{t=1}^{T} \sum_{n=1}^{N} \Big(I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n}) \cdot \ln m_j \\
& + \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j) \cdot \ln(1 - m_j) \Big)
\end{aligned}
$$

Computing the Most Likely Map

“beam *n* ends in cell *j*”

$$m^\star = \operatorname{argmax}_m \sum_{j=1}^{J} \sum_{t=1}^{T} \sum_{n=1}^{N} \Big(I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n}) \cdot \ln m_j$$

$$+ \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j) \cdot \ln(1 - m_j) \Big)$$

“beam *n* traversed cell *j*”

Computing the Most Likely Map

“beam *n* ends in cell *j*”

$$m^\star = \operatorname{argmax}_m \sum_{j=1}^{J} \sum_{t=1}^{T} \sum_{n=1}^{N} \Big(I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n}) \cdot \ln m_j$$

$$+ \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j) \cdot \ln(1 - m_j) \Big)$$

“beam *n* traversed cell *j*”

Define

$$\alpha_j = \sum_{t=1}^{T} \sum_{n=1}^{N} I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n})$$

$$\beta_j = \sum_{t=1}^{T} \sum_{n=1}^{N} \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j)$$

Meaning of α*j* and β*j*

$$\alpha_j = \sum_{t=1}^{T} \sum_{n=1}^{N} I(f(x_t, n, z_{t,n}) = j) \cdot (1 - \zeta_{t,n})$$

Corresponds to the number of times a beam that is not a maximum range beam ended in cell *j* ($hits(j)$)

$$\beta_j = \sum_{t=1}^{T} \sum_{n=1}^{N} \sum_{k=0}^{z_{t,n}-1} I(f(x_t, n, k) = j)$$

Corresponds to the number of times a beam traversed cell *j* without ending in it ($misses(j)$)

Computing the Most Likely Map

Accordingly, we get

$$\mathrm{m}^\star = \operatorname{argmax}_m \textstyle\sum_{j=1}^{J} \Big(\alpha_j \ln m_j + \beta_j \ln(1 - m_j) \Big)$$

As the mj's are independent we can maximize this sum by maximizing it for every j

If we set

$$\frac{\partial}{\partial m_j} = \frac{\alpha_j}{m_j} - \frac{\beta_j}{1 - m_j} = 0$$

we obtain

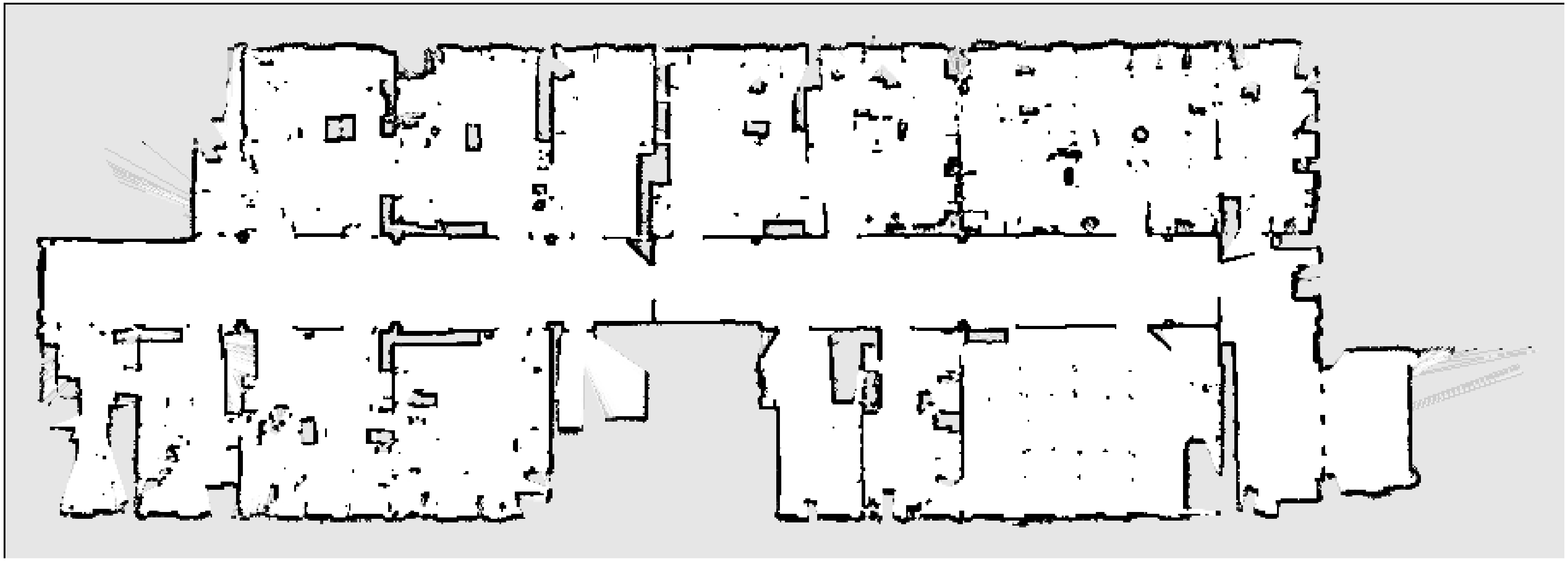
$$m_j = \frac{\alpha_j}{\alpha_j + \beta_j}$$

Computing the most likely map reduces to counting how often a cell has reflected a measurement and how often the cell was traversed by a beam.

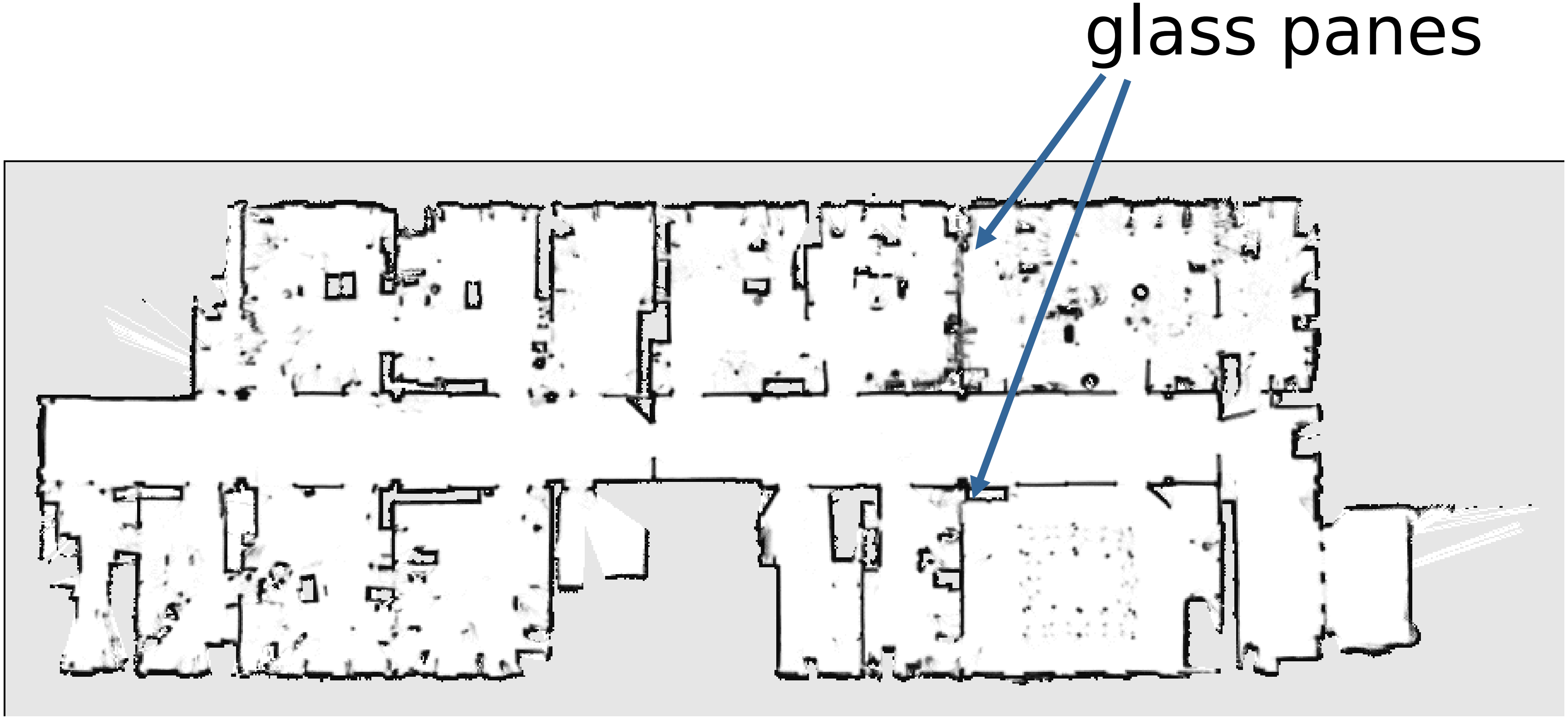
Difference between Occupancy Grid Maps and Counting

- The counting model determines how often a cell reflects a beam.
- The occupancy model represents whether or not a cell is occupied by an object.
- Although a cell might be occupied by an object, the reflection probability of this object might be very small.

Example Occupancy Map



Example Reflection Map



Example

- Out of n beams only 60% are reflected from a cell and 40% intercept it without ending in it.
- Accordingly, the reflection probability will be 0.6.
- Suppose *p(occ | z)* = 0.55 when a beam ends in a cell and *p(occ | z)* = 0.45 when a beam traverses a cell without ending in it.
- Accordingly, after *n* measurements we will have

$$\left(\frac{0.55}{0.45}\right)^{n*0.6} * \left(\frac{0.45}{0.55}\right)^{n*0.4} = \left(\frac{11}{9}\right)^{n*0.6} * \left(\frac{11}{9}\right)^{-n*0.4} = \left(\frac{11}{9}\right)^{n*0.2}$$

- Whereas the reflection map yields a value of 0.6, the occupancy grid value converges to 1 as n increases.

Summary (1)

- Grid maps are a popular environment model
- Occupancy grid maps discretize the space into independent cells
- Each cell is a binary random variable estimating if the cell is occupied
- Static state binary Bayes filter per cell
- Mapping with known poses is easy
- Log odds model is fast to compute

Summary (2)

- Reflection probability maps are an alternative representation
- They store in each cell the probability that a beam is reflected by this cell
- Given the described sensor model, counting yields the maximum likelihood model