Introduction to Mobile Robotics

Compact Course on Linear Algebra

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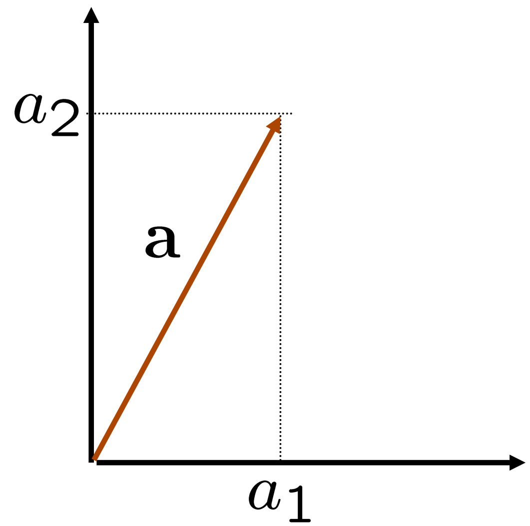
Vectors

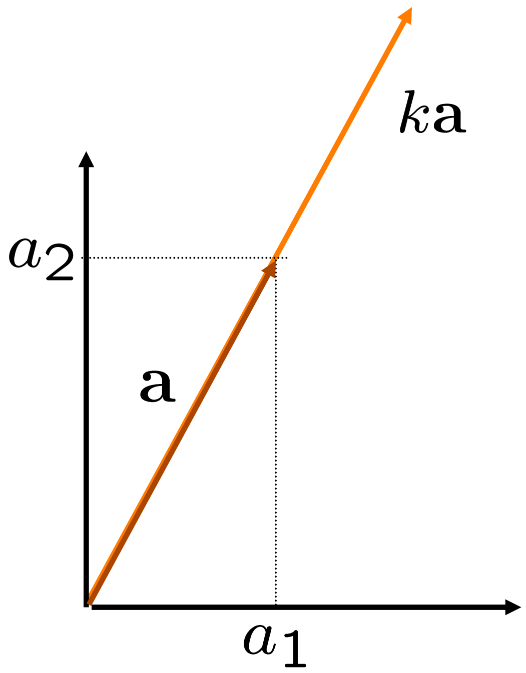
- Arrays of numbers
- Vectors represent a point in a *n* dimensional space

$$(a_1) \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Vectors: Scalar Product

- Scalar-Vector Product $k\mathbf{a}$
- Changes the length of the vector, but **not** its direction

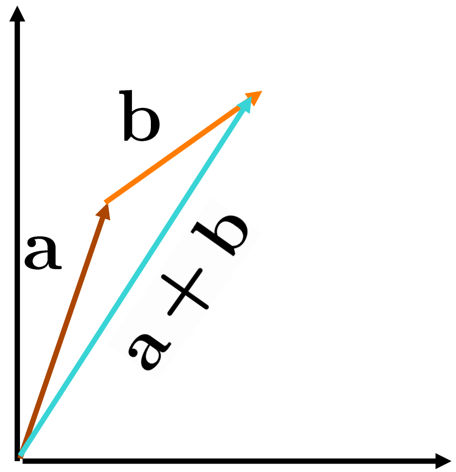


Vectors: Sum

- Sum of vectors (is commutative)

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Can be visualized as "chaining" the vectors.

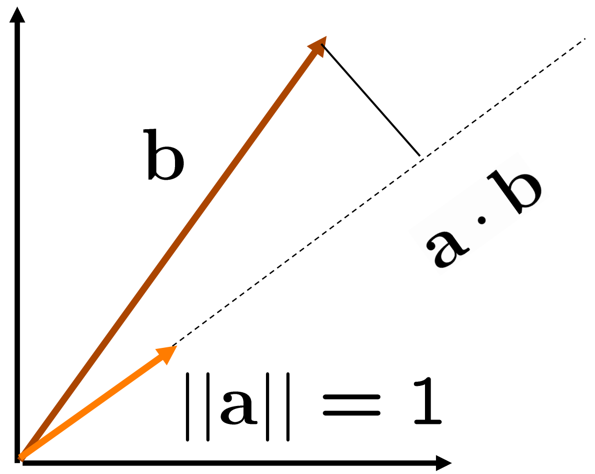


Vectors: Dot Product

- Inner product of vectors (is a scalar)

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \sum_i a_i b_i$$

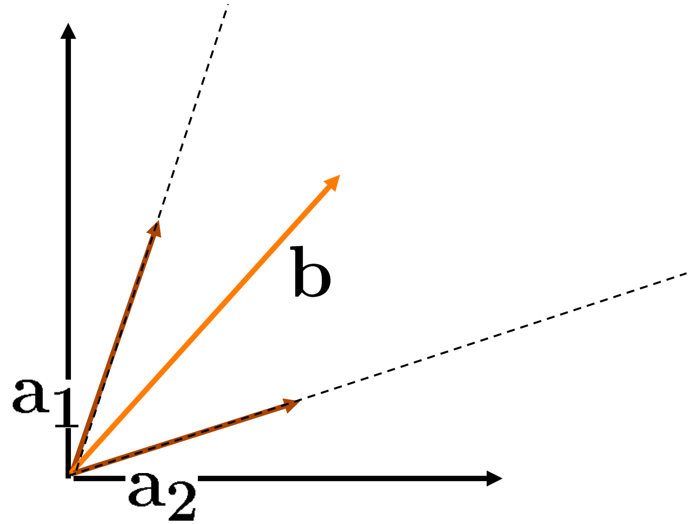
- If one of the two vectors, e.g. $\mathbf{a}$, has $||\mathbf{a}|| = 1$ the inner product $\mathbf{a} \cdot \mathbf{b}$ returns the length of the projection of $\mathbf{b}$ along the direction of $\mathbf{a}$



- If $\mathbf{a} \cdot \mathbf{b} = 0$, the two vectors are **orthogonal**

Vectors: Linear (In)Dependence

- A vector $\mathbf{b}$ is **linearly dependent** from $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ if $\mathbf{b} = \sum_i k_i \mathbf{a}_i$
- In other words, if $\mathbf{b}$ can be obtained by summing up the $\mathbf{a}_i$ properly scaled
- If there exist no $\{k_i\}$ such that $\mathbf{b} = \sum_i k_i \mathbf{a}_i$ then $\mathbf{b}$ is independent from $\{\mathbf{a}_i\}$



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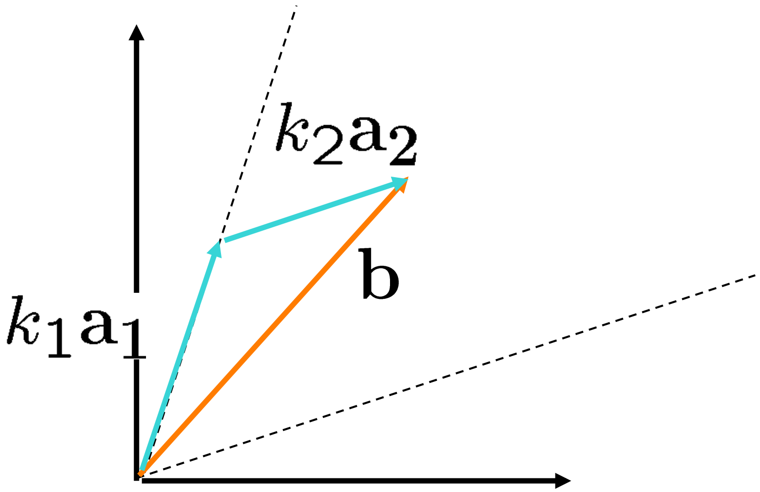


Matrices

- A matrix is written as a table of values

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \qquad A : \underset{\substack{\uparrow \\ \text{rows}}}{n} \times \underset{\substack{\uparrow \\ \text{columns}}}{m}$$

- **1st index** refers to the **row**
- **2nd index** refers to the **column**
- Note: a d-dimensional vector is equivalent to a dx1 matrix

Matrices as Collections of Vectors

- Column vectors

$$\begin{pmatrix} \mathbf{a}_{*1} & \mathbf{a}_{*2} & \cdots & \mathbf{a}_{*m} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \\ a_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \\ a_{nm} \end{matrix}} \end{pmatrix}$$

Matrices as Collections of Vectors

- Row vectors

$$\mathbf{A} = \begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{pmatrix} \begin{matrix} \longrightarrow \\ \longrightarrow \\ \\ \longrightarrow \end{matrix} \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{*n}^T \end{pmatrix}$$

Important Matrices Operations

- Multiplication by a scalar
- Sum (commutative, associative)
- Multiplication by a vector
- Product (not commutative)
- Inversion (square, full rank)
- Transposition

Scalar Multiplication & Sum

- In the scalar multiplication, every element of the vector or matrix is multiplied with the scalar
- The sum of two vectors is a vector consisting of the pair-wise sums of the individual entries
- The sum of two matrices is a matrix consisting of the pair-wise sums of the individual entries

Matrix Vector Product

- The *i^th* component of $\mathbf{Ab}$ is the dot product $\mathbf{a}_{i*}^T \cdot \mathbf{b}$
- The vector $\mathbf{Ab}$ is linearly dependent from the column vectors $\{\mathbf{a}_{*i}\}$ with coefficients $\{b_i\}$

$$\mathbf{Ab} = \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{n*}^T \end{pmatrix} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \cdot \mathbf{b} \\ \mathbf{a}_{2*}^T \cdot \mathbf{b} \\ \vdots \\ \mathbf{a}_{n*}^T \cdot \mathbf{b} \end{pmatrix} = \sum_k \mathbf{a}_{*k} \cdot b_k$$

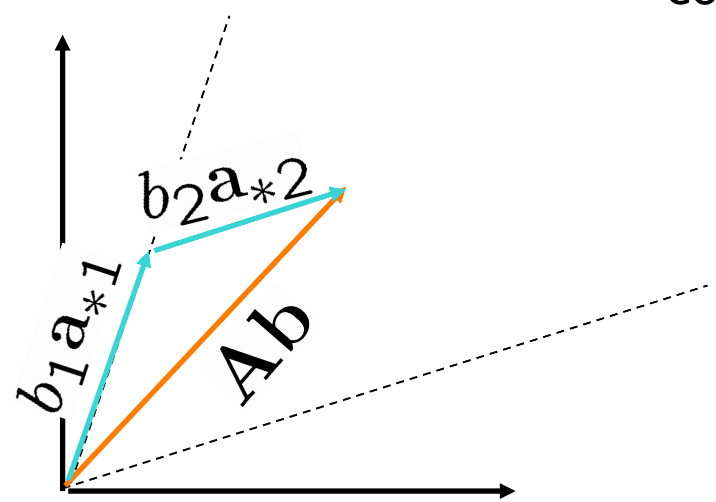
row vectors

column vectors

Matrix Vector Product

- If the column vectors of $\mathbf{A}$ represent a reference system, the product $\mathbf{A}\mathbf{b}$ computes the global transformation of the vector $\mathbf{b}$ according to $\{\mathbf{a}_{*i}\}$

column vectors



Matrix Matrix Product

- Can be defined through
  - the dot product of row and column vectors
  - the linear combination of the columns of ***A*** scaled by the coefficients of the columns of ***B***

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A}\mathbf{B} \\
&= \begin{pmatrix}
\mathbf{a}_{1*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*m} \\
\mathbf{a}_{2*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*m} \\
\vdots & & & \\
\mathbf{a}_{n*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*m}
\end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}\mathbf{b}_{*1} & \mathbf{A}\mathbf{b}_{*2} & \ldots \mathbf{A}\mathbf{b}_{*m} \end{pmatrix}
\end{aligned}
$$

column vectors

Matrix Matrix Product

- If we consider the second interpretation, we see that the columns of ***C*** are the "global transformations" of the columns of ***B*** through ***A***
- All the interpretations made for the matrix vector product hold

$$\begin{aligned} \mathbf{C} &= \mathbf{A}\mathbf{B} \\ &= \left(\mathbf{A}\mathbf{b}_{*1} \;\; \mathbf{A}\mathbf{b}_{*2} \;\; \ldots \mathbf{A}\mathbf{b}_{*m} \right) \\ \mathbf{c}_{*i} &= \mathbf{A}\mathbf{b}_{*i} \end{aligned}$$

column vectors

Linear Systems (1)

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Interpretations:

- A set of linear equations
- A way to find the coordinates ***x*** in the reference system of ***A*** such that ***b*** is the result of the transformation of ***Ax***
- Solvable by Gaussian elimination

Linear Systems (2)

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Notes:

- Many efficient solvers exit, e.g., conjugate gradients, sparse Cholesky decomposition
- One can obtain a reduced system ***(A′, b′)*** by considering the matrix ***(A, b)*** and suppressing all the rows which are linearly dependent
- Let ***A'x=b'*** the reduced system with ***A':****n'xm* and ***b'****:n'x1* and rank **A'** = *min(n',m)* rows columns
- The system might be either over-constrained *(n′>m)* or under-constrained *(n′<m)*

Over-Constrained Systems

- "More (indep) equations than variables"
- An over-constrained system does not admit an **exact solution**
- However, if *rank* ***A'*** = *cols(**A**)* one often computes a **minimum norm solution**

$$\mathbf{x} = \underset{\mathbf{x}}{\operatorname{argmin}} \, ||\mathbf{A}'\mathbf{x} - \mathbf{b}'||$$

Note: rank = Maximum number of linearly independent rows/columns

Under-Constrained Systems

- "More variables than (indep) equations"
- The system is **under-constrained** if the number of linearly independent rows of ***A'*** is smaller than the dimension of ***b'***
- An under-constrained system admits infinite solutions
- The degree of these infinite solutions is *cols*(***A'***) - *rows*(***A'***)

Inverse

$$\mathrm{AB} = \mathrm{I}$$

- If A is a square matrix of full rank, then there is a unique matrix $\boldsymbol{B}=\boldsymbol{A}^{-1}$ such that $\boldsymbol{AB}=\boldsymbol{I}$ holds
- The i^{th} row of $\boldsymbol{A}$ is and the j^{th} column of $\boldsymbol{A}^{-1}$ are:
  - orthogonal (if $i \neq j$)
  - or their dot product is 1 (if $i = j$)

Matrix Inversion

$$\mathbf{AB} = \mathbf{I}$$

- The i^{th} column of $\boldsymbol{A^{-1}}$ can be found by solving the following linear system:

$$\mathbf{Aa^{-1}}_{*i} = \mathbf{i}_{*i}$$

This is the i^{th} column of the identity matrix

Rank

- **Maximum** number of linearly independent rows (columns)
- Dimension of the **image** of the transformation $f(\mathbf{x}) = A\mathbf{x}$

- When A is $m \times n$ we have
  - $\mathrm{rank}(A) \geq 0$ and the equality holds iff A is the null matrix
  - $\mathrm{rank}(A) \leq \min(m, n)$
  - $f(\mathbf{x})$ is **injective** iff $\mathrm{rank}(A) = n$
  - $f(\mathbf{x})$ is **surjective** iff $\mathrm{rank}(A) = m$
  - if $m = n$, $f(\mathbf{x})$ is **bijective** and A is **invertible** iff $\mathrm{rank}(A) = n$

- Computation of the rank is done by
  - Gaussian elimination on the matrix
  - Counting the number of non-zero rows

Determinant (det)

- Only defined for **square matrices**
- The inverse of $\mathbf{A}$ exists if and only if $det(\mathbf{A}) \neq 0$
- For 2×2 matrices:

  Let $\mathbf{A} = [a_{ij}]$ and $|\mathbf{A}| = det(\mathbf{A})$, then

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- For 3×3 matrices the Sarrus rule holds:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11}$$

Determinant

- For **general** $n \times n$ matrices?

  Let $\mathbf{A}_{ij}$ be the submatrix obtained from $\mathbf{A}$
  by deleting the i-th row and the j-th column

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & 3 & 4 & -1 \\ -5 & 8 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

  Rewrite determinant for 3×3 matrices:

$$\begin{aligned} det(\mathbf{A}^{3\times3}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11} \\ &= a_{11} \cdot det(\mathbf{A}_{11}) - a_{12} \cdot det(\mathbf{A}_{12}) + a_{13} \cdot det(\mathbf{A}_{13}) \end{aligned}$$

Determinant

- For **general** $n \times n$ matrices?

$$\begin{aligned} det(\mathbf{A}) &= a_{11}det(\mathbf{A}_{11}) - a_{12}det(\mathbf{A}_{12}) + ... + (-1)^{1+n}a_{1n}det(\mathbf{A}_{1n}) \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}det(\mathbf{A}_{1j}) \end{aligned}$$

Let $\mathbf{C}_{ij} = (-1)^{i+j}det(\mathbf{A}_{ij})$ be the *(i,j)*-cofactor, then

$$\begin{aligned} det(\mathbf{A}) &= a_{11}\mathbf{C}_{11} + a_{12}\mathbf{C}_{12} + ... + a_{1n}\mathbf{C}_{1n} \\ &= \sum_{j=1}^{n} a_{1j}\mathbf{C}_{1j} \end{aligned}$$

This is called the **cofactor expansion** across the first row

Determinant

- **Problem:** Take a 25 x 25 matrix (which is considered small). The cofactor expansion method requires n! multiplications. For n = 25, this is 1.5 x 10^25 multiplications for which a today supercomputer would take **500,000 years**.

- There are **much faster methods**, namely using **Gauss elimination** to bring the matrix into triangular form.

$$\mathbf{A} = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{bmatrix} \qquad det(\mathbf{A}) = \prod_{i=1}^{n} d_i$$

Because for **triangular matrices** the determinant is the product of diagonal elements

Determinant: Properties

- **Row operations** ($\mathbf{A}$ is still a $n \times n$ square matrix)
  - If $\mathbf{B}$ results from $\mathbf{A}$ by interchanging two rows, then $det(\mathbf{B}) = -det(\mathbf{A})$
  - If $\mathbf{B}$ results from $\mathbf{A}$ by multiplying one row with a number c, then $det(\mathbf{B}) = c \cdot det(\mathbf{A})$
  - If $\mathbf{B}$ results from $\mathbf{A}$ by adding a multiple of one row to another row, then $det(\mathbf{B}) = det(\mathbf{A})$

- **Transpose**: $det(\mathbf{A}^T) = det(\mathbf{A})$

- **Multiplication**: $det(\mathbf{A} \cdot \mathbf{B}) = det(\mathbf{A}) \cdot det(\mathbf{B})$

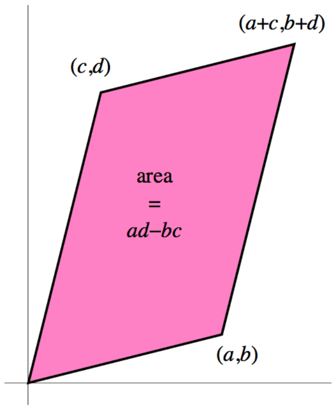
- Does **not** apply to addition! $det(\mathbf{A} + \mathbf{B}) \neq det(\mathbf{A}) + det(\mathbf{B})$

Determinant: Applications

- Compute **Eigenvalues:**
  Solve the characteristic polynomial $det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$

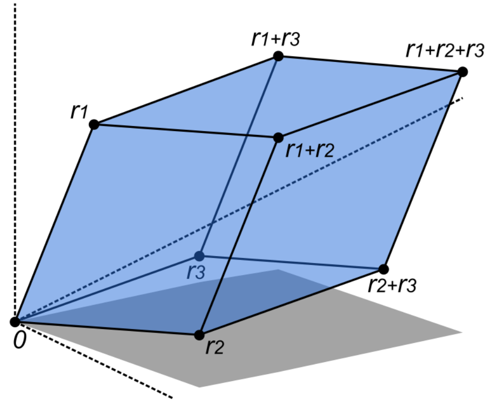
- **Area** and **Volume:** $\text{area} = |det(\mathbf{A})|$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(r_i is i-th row)



Orthonormal Matrix

- A matrix Q is **orthonormal** iff its column (row) vectors represent an **orthonormal** basis

$$q_{*i}^T \cdot q_{*j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}, \forall \ i, j$$

- As linear transformation, it is **norm** preserving

- Some properties:
  - The transpose is the inverse $QQ^T = Q^TQ = I$
  - Determinant has unity norm ($\pm$ 1)

$$1 = det(I) = det(Q^TQ) = det(Q)det(Q^T) = det(Q)^2$$

Rotation Matrix

- A Rotation matrix is an orthonormal matrix with det =+1
  - 2D Rotations $$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$
  - 3D Rotations along the main axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- **IMPORTANT: Rotations are not commutative**

$$R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \quad R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \quad R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Matrices to Represent Affine Transformations

- A general and easy way to describe a 3D transformation is via matrices

Translation Vector

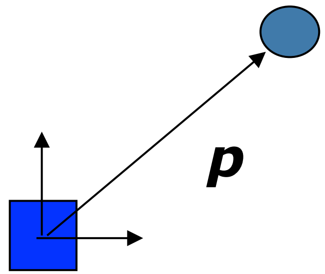
$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^{T} & -\mathbf{R}^{T}\mathbf{t} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} \mathbf{t} \\ \mathbf{1} \end{pmatrix}$$

Rotation Matrix

- Takes naturally into account the non-commutativity of the transformations
- Homogeneous coordinates

Combining Transformations

- A simple interpretation: chaining of transformations (represented as homogeneous matrices)
  - Matrix ***A*** represents the pose of a **robot** in the space
  - Matrix ***B*** represents the position of a sensor on the robot
  - The **sensor** perceives an **object** at a given location ***p***, in its own frame [the sensor has no clue on where it is in the world]
  - Where is the object in the global frame?



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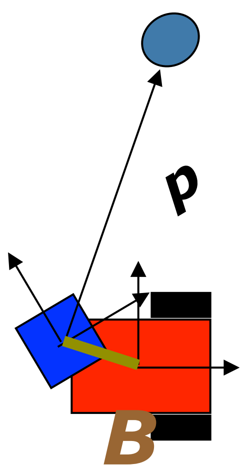

Bp gives the pose of the object wrt the robot

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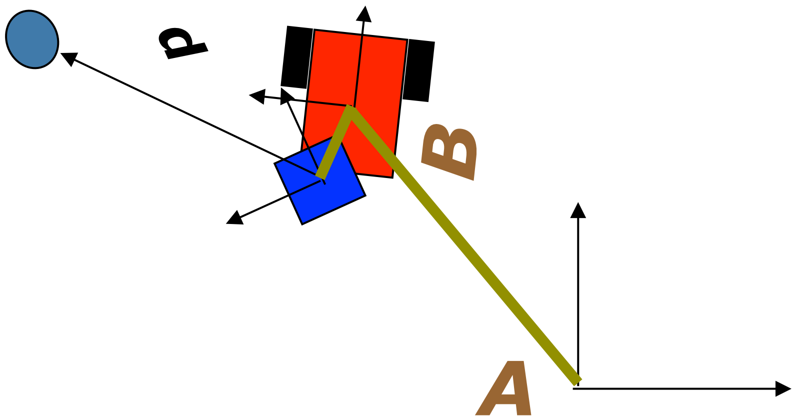


Bp gives the pose of the object wrt the robot

ABp gives the pose of the object wrt the world

Positive Definite Matrix

- The analogous of positive number
- Definition $M > 0$ iff $z^T M z > 0 \forall z \neq 0$
- Example
  - $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0$

Positive Definite Matrix

- Properties
  - **Invertible**, with positive definite inverse
  - All real **eigenvalues** > 0
  - **Trace** is > 0
  - **Cholesky** decomposition $A = LL^T$

Jacobian Matrix

- It is a **non-square matrix** $n \times m$ in general
- Given a vector-valued function

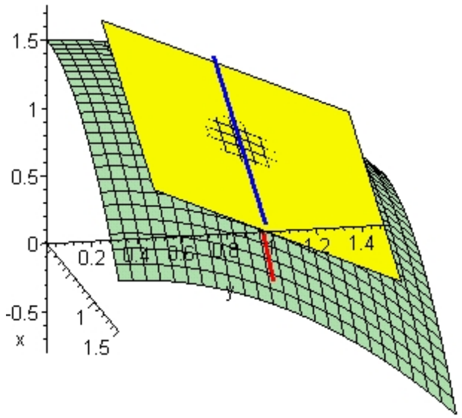
$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

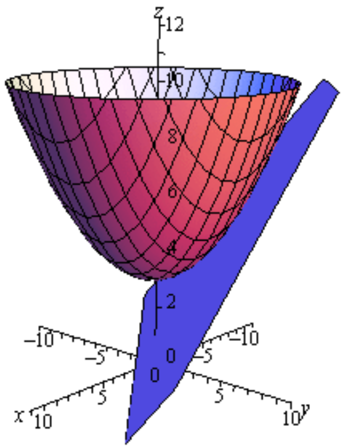
- Then, the **Jacobian matrix** is defined as

$$\mathbf{F_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It is the orientation of the **tangent plane** to the vector-valued function at a given point





- **Generalizes the gradient** of a scalar valued function

Further Reading

- A "quick and dirty" guide to matrices is the Matrix Cookbook available at:

  http://matrixcookbook.com