2.1_PoseIn2D

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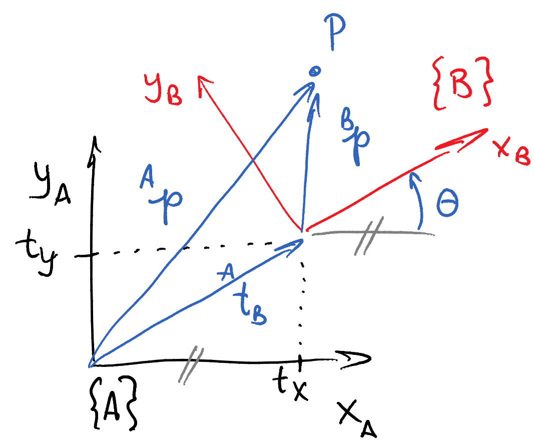
2.1. Pose in 2D

- ^A xi_B = (x,y,th)
- R = [cos(th) -sin(th); sin(th) cos(th)]
- Example ^A p = ^A R_B * ^B p
- R \in SO(2) = Special Orthogonal Group : (-)R = R' and det (R) = +1
- Also note that R(-th) = (-)R(th) so rotation ccw is the inverse of rotation cw
- Why only one parameter, th? 4 elements in a 2x2 matrix
    - Col (row) orthogonality ==> one constraint
    - Col (row) normality ==> two constraints
    - Therefore only 1 dof, so only one parameter is needed
- Accounting for translation - eqs.(2.7 - 2.10)
- Define SE(2) with elements known as homogeneous transformations
- Notation xi(x,y,th) ~ htform
- Orange box. Give result of composition and inverse.

- Introduce matlab toolbox function, se2(x,y,th)
- Matlab example in fig 2.8
- Introduce e2h (euclidean to homogeneous) and h2e
- Introduce homtrans



The pose of a rigid body is defined by three variables :

$$\begin{bmatrix} (^A t_B)_x \\ (^A t_B)_y \\ \hdashline \theta \end{bmatrix} = \begin{bmatrix} ^A t_B \\ \hdashline \theta \end{bmatrix}$$

where $^A t_B \in \mathbb{R}^2$ = space of 2-dimensional vectors
a.k.a., Euclidean 2-space

$\theta \in S^1$ = $[0, 2\pi)$
a.k.a., the circle of dimension 1

Translate origin of {B} by t relative to {A}, i.e. $^A t_B$

Rotate {B} about its new origin by θ.

Corke derives the 3-by-3 homogeneous transformation matrix that maps a point from {B} to {A}.

that maps a point from $\{B\}$ to $\{A\}$.

$$
{}^{A}\xi_{B}(t,\theta) \sim {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}t_{B} \\ 0 \quad 0 & 1 \end{bmatrix}_{(3\times 3)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & ({}^{A}t_{B})_{x} \\ \sin(\theta) & \cos(\theta) & ({}^{A}t_{B})_{y} \\ 0 & 0 & 1 \end{bmatrix}
$$

${}^{A}\hat{x}_{B}$ direction ${}^{A}\hat{y}_{B}$ direction origin of $\{B\}$

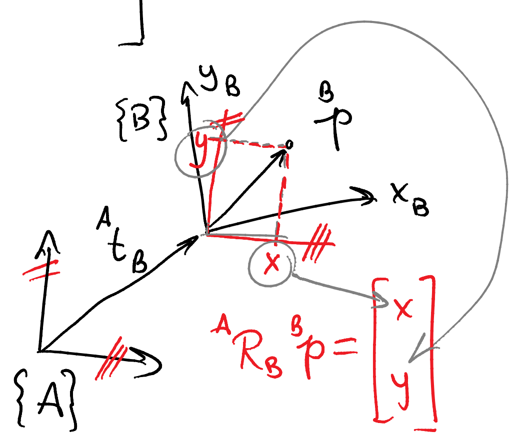
express in $\{A\}$ and written in homogeneous form

$$
{}^{A}R_{B} = \left[{}^{A}\hat{x}_{B} \;\vdots\; {}^{A}\hat{y}_{B} \right]_{(2\times 2)}
$$

$$
\boxed{{}^{A}\tilde{p} = {}^{A}T_{B}\, {}^{B}\tilde{p}}
$$

where $\tilde{p} \triangleq \begin{bmatrix} p \\ \hdashline 1 \end{bmatrix}_{(3\times 1)}$ is the homogeneous form of p.

$$
\begin{bmatrix} {}^{A}p \\ \hdashline 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}t_{B} \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}p \\ \hdashline 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B}\, {}^{B}p + {}^{A}t_{B} \\ 1 \end{bmatrix}
$$



${}^{A}R_{B}\, {}^{B}p$ expresses ${}^{B}p$ in $\{A\}$

${}^{A}t_{B}$ adds the displacement.

${}^{A}R_{B}$ is needed to write the ${}^{B}p$ vector in $\{A\}$ so it can be added to the other vector in $\{A\}$, namely ${}^{A}t$.

Special Euclidean Group $SE(2) = SO(2) \times \mathbb{R}^2$

$\mathbb{R}^2$ = Euclidean group = 2-dimension vector space.

"$\times$" = Cartesian product or set product

each element of $SO(2)$ is combined with each element of $\mathbb{R}(2)$

$R \in SO(2)$, SO stands for Special Orthogonal group of 2x2 matrices

$$A_{(2\times2)} \in SO(2) \text{ iff } A^T = A^{-1} \Rightarrow A^T A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{(2\times2)}$$

and $\det(A) = 1$

Let a_1 and a_2 be columns of A, ie, $A = [a_1\ a_2]$.

$$A^T A^{-1} = I_{(2\times2)} \Rightarrow \left.\begin{aligned} \|a_1\| &= 1 \\ \|a_2\| &= 1 \\ a_1^T a_2 &= 0 \end{aligned}\right\} \text{3 constraints}$$

A (2x2) matrix has 4 elements.

3 constraints $\Rightarrow$ A has only 1 free variable!

$\therefore$ Planar orientation can be represented with one variable!

SO(2) is a nice way to represent 2d orientation because it handles wrap-around

Combining an element of SO(2) and $\mathbb{R}^2$ yields an element of SE(2).

Since SO(2) is 1-dimensional and $\mathbb{R}^2$ is 2-dimensional, SE(2) is 3-dimensional

Therefore 3 variables are needed to define a planar pose.

Each element of SE(2) is said to represent a rigid body displacement in the plane.

A couple special results:

$$\ominus {}^A\xi_B \sim ({}^A T_B)^{-1} = \begin{bmatrix} {}^A R_B & {}^A t_B \\ 0 \; 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} {}^A R_B^T & -{}^B R_A^T \, {}^A t_B \\ 0 \; 0 & 1 \end{bmatrix}$$

$$ {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} \sim {}^{A}T_{B}\,{}^{B}T_{C} = {}^{A}T_{C} = \left[\begin{array}{c:c} {}^{A}R_{B}\,{}^{B}R_{C} & {}^{A}t_{B} + {}^{A}R_{B}\,{}^{B}t_{C} \\ \hdashline 0 \quad 0 & 1 \end{array}\right]_{(3\times 3)} $$

Given $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$, one can always find θ by:

$$ \theta = \text{atan2}(r_{21}, r_{11}) $$

Rotations in the plane commute, i.e. $R_1 R_2 = R_2 R_1$

Matlab functions:

se2(x,y,θ) $\qquad$ $se2(1,2,\pi/6) = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

axis

trplot2

hold on

plot_point

inv

e2h

h2e

homtrans