Foundations of Artificial Intelligence

10. Satisfiability and Model Construction

Davis-Putnam-Logemann-Loveland Procedure, Phase Transitions, GSAT

Wolfram Burgard, Bernhard Nebel, and Martin Riedmiller

Albert-Ludwigs-Universität Freiburg

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Contents

Motivation

- Usually:
  - Given: A logical theory (set of propositions)
  - Question: Does a proposition logically follow from this theory?
  - Reduction to unsatisfiability, which is coNP-complete (complementary to NP problems)
- Sometimes:
  - Given: A logical theory
  - Wanted: Model of the theory
  - Example: Configurations that fulfill the constraints given in the theory
  - Can be "easier" because it is enough to find one model

The DPLL Procedure

DPLL Function

Given a set of clauses Δ defined over a set of variables Σ, return "satisfiable" if Δ is satisfiable. Otherwise return "unsatisfiable".

1. If $\Delta = \emptyset$ return "satisfiable"
2. If $\Box \in \Delta$ return "unsatisfiable"
3. Unit-propagation Rule: If Δ contains a unit-clause C, assign a truth-value to the variable in C that satisfies C, simplify Δ to Δ' and return $\mathrm{DPLL}(\Delta')$.
4. Splitting Rule: Select from Σ a variable v which has not been assigned a truth-value. Assign one truth value t to it, simplify Δ to Δ' and call $\mathrm{DPLL}(\Delta')$
   a. If the call returns "satisfiable", then return "satisfiable".
   b. Otherwise assign *the other* truth-value to v in Δ, simplify to Δ'' and return $\mathrm{DPLL}(\Delta'')$.

Example (1)

$$\Delta = \{\{a, b, \neg c\}, \{\neg a, \neg b\}, \{c\}, \{a, \neg b\}\}$$

1. Unit-propagation rule: $c \mapsto T$
   $\{\{a, b\}, \{\neg a, \neg b\}, \{a, \neg b\}\}$
2. Splitting rule:

2a. $a \mapsto F$
$\{\{b\}, \{\neg b\}\}$

3a. Unit-propagation rule: $b \mapsto T$
$\{\square\}$

2b. $a \mapsto T$
$\{\{\neg b\}\}$

3b. Unit-propagation rule: $b \mapsto F$
$\{\}$

Example (2)

$$\Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\}$$

1. Unit-propagation rule: $d \mapsto T$
   $\{\{a, \neg b, \neg c\}, \{b\}, \{c\}\}$
2. Unit-propagation rule: $b \mapsto T$
   $\{\{a, \neg c\}, \{c\}\}$
3. Unit-propagation rule: $c \mapsto T$
   $\{\{a\}\}$
4. Unit-propagation rule: $a \mapsto T$
   $\{\}$

Properties of DPLL

- DPLL is complete, correct, and guaranteed to terminate.
- DPLL constructs a model, if one exists.
- In general, DPLL requires exponential time (splitting rule!)
- DPLL is polynomial on Horn clauses, i.e., clauses with at most one positive literal

$$\neg A_1, \vee \ldots \vee \neg A_n \vee B \Leftrightarrow \bigwedge_i A_i \Rightarrow B$$

$\rightarrow$ *Heuristics* are needed to determine which variable should be instantiated next and which value should be used.

$\rightarrow$ In all SAT competitions so far, DPLL-based procedures have shown the best performance.

DPLL on Horn Clauses (1)

Note:

1. The simplifications in DPLL on Horn clauses always generate Horn clauses
2. A set of Horn clauses without unit clauses is satisfiable
   - *All clauses have at least one negative literal*
   - *Assign $false$ to all variables*
3. If the first sequence of applications of the unit propagation rule in DPLL does not lead to the empty clause, a set of Horn clauses without unit clauses is generated (which is satisfiable according to 2.)

DPLL on Horn Clauses (2)

4. Although a set of Horn clauses without a unit clause is satisfiable, DPLL may not immediately recognize it

   a. If DPLL assigns *false* to a variable, this cannot lead to an unsatisfiable set and after a sequence of unit propagations we are in the same situation as in 4.

   b. If DPLL assigns *true*, then we may get an empty clause - perhaps after unit propagation (and have to backtrack) - or the set is still satisfiable and we are in the same situation as in 4.

DPLL on Horn Clauses (3)

In summary:

1. DPLL executes a sequence of unit propagation steps resulting in
   - an empty clause or
   - a set of Horn clauses without a unit clause, which is satisfiable
2. In the latter case, DPLL proceeds by choosing for one variable:
   - *false*, which does not change the satisfiability
   - *true*, which either
     - leads to an immediate contradiction (after unit propagation) and backtracking or
     - does not change satisfiabilty

$\rightarrow$ Run time is *polynomial* in the number of variables.

How Good is DPLL in the Average Case?

- We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.
- This is clearly also true for the DPLL-procedure.
  $\rightarrow$ Couldn't we do better in the average case?
- For CNF-formulae in which the probability for a positive appearance, negative appearance and non-appearance in a clause is 1/3, DPLL needs on average quadratic time (Goldberg 79)!
  $\rightarrow$ The probability that these formulae are satisfiable is, however, very high.

Phase Transitions . . .

Conversely, we can, of course, try to identify hard to solve problem instances.

Cheeseman et al. (IJCAI-91) came up with the following plausible conjecture:

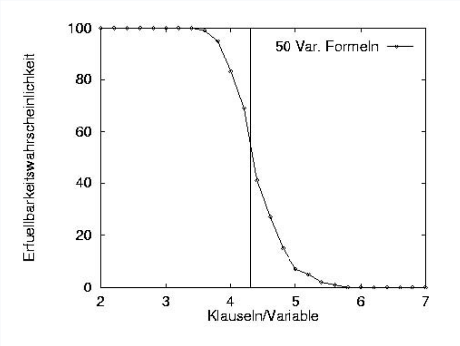
> All NP-complete problems have at least *one order* parameter and the hard to solve problems are around a critical value of this order parameter. This critical value (a phase transition) separates one region from another, such as over-constrained and under-constrained regions of the problem space.

Confirmation for graph coloring and Hamilton path . . . later also for other NP-complete problems.

Phase Transitions with 3-SAT

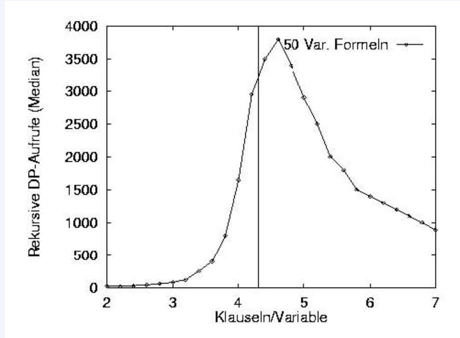
Constant clause length model (Mitchell et al., AAAI-92):
Clause length k is given. Choose variables for every clause k and use the complement with probability 0.5 for each variable.

Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:



Empirical Difficulty

The Davis-Putnam (DPLL) Procedure shows extreme runtime peaks at the phase transition:



Note: Hard instances can exist even in the regions of the more easily satisfiable/unsatisfiable instances!

Notes on the Phase Transition

- When the probability of a solution is close to 1 (under-constrained), there are many solutions, and the first search path of a backtracking search is usually successful.
- If the probability of a solution is close to 0 (over-constrained), this fact can usually be determined early in the search.
- In the phase transition stage, there are many near successes ("close, but no cigar")

  → (limited) possibility of predicting the difficulty of finding a solution based on the parameters

  → (search intensive) benchmark problems are located in the phase region (but they have a special structure)

Local Search Methods for Solving Logical Problems

In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can "solve" much large instances this way.

Standard process for optimization problems: Local Search

- Based on a (random) configuration
- Through local modifications, we hope to produce better configurations

  → Main problem: local maxima

Dealing with Local Maxima

As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.

But local search seems inappropriate, considering we want to find a global maximum (all constraints/clauses satisfied).

By restarting and/or injecting noise, we can often escape local maxima.

Actually: Local search performs very well for finding satisfying assignments of CNF formulae (even without injecting noise).

GSAT

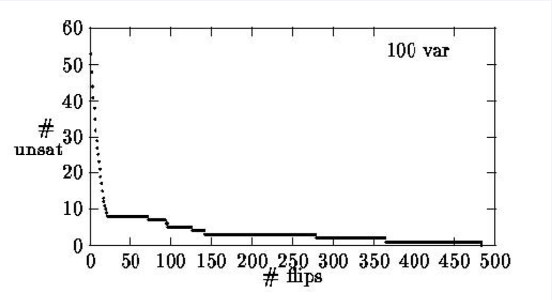
Procedure GSAT

INPUT: a set of clauses α, MAX-FLIPS, and MAX-TRIES
OUTPUT: a satisfying truth assignment of α, if found

begin
  for $i := 1$ to MAX-TRIES
    $T :=$ a randomly-generated truth assignment
    for $j := 1$ to MAX-FLIPS
      if T satisfies α **then return** T
      $v :=$ a propositional variable such that a change in its truth assignment gives the largest increase in the number of clauses of α that are satisfied by T
      $T := T$ with the truth assignment of v reversed
    end for
  end for
  return "no satisfying assignment found"
end

The Search Behavior of GSAT

- In contrast to normal local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.



State of the Art

- SAT competitions since beginning of the 90s
- Current SAT competitions (http://www.satcompetition.org/):
  In 2010:
  - Largest "industrial" instances: $>$ 1,000,000 literals
- Complete solvers are as good as randomized ones on handcrafted and industrial problem

Concluding Remarks

- DPLL-based SAT solvers prevail:
  - Very efficient implementation techniques
  - Good branching heuristics
  - Clause learning
- Incomplete randomized SAT-solvers
  - are good (in particular on random instances)
  - but there is no dramatic increase in size of what they can solve
  - parameters are difficult to adjust