Foundations of Artificial Intelligence

5. Constraint Satisfaction Problems

CSPs as Search Problems, Solving CSPs, Problem Structure

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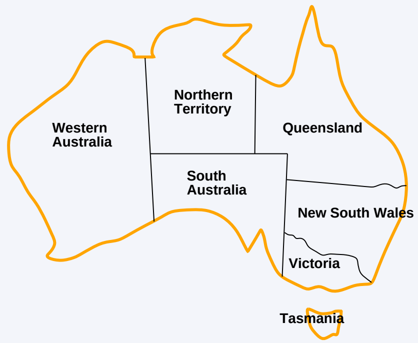
Constraint Satisfaction Problems

- A Constraint Satisfaction Problems (CSP) consists of
  - a set of variables $\{X_1, X_2, \ldots, X_n\}$ to which
  - values $\{d_1, d_2, ..., d_k\}$ can be assigned
  - such that a set of constraints over the variables is respected
- A CSP is solved by a variable assignment that satisfies all given constraints.
- In CSPs, states are explicitly represented as variable assignments. CSP search algorithms take advantage of this structure.
- The main idea is to exploit the constraints to eliminate large portions of search space.
- *Formal representation language* with associated general inference algorithms

Example: Map-Coloring

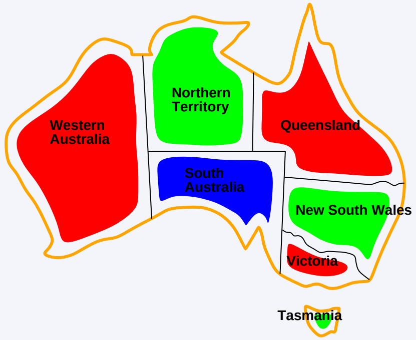


- Variables: WA, NT, SA, Q, NSW, V, T
- Values: $\{red, green, blue\}$
- Constraints: adjacent regions must have different colors, e.g., $NSW \neq V$

Australian Capital Territory (ACT) and Canberra (inside NSW)

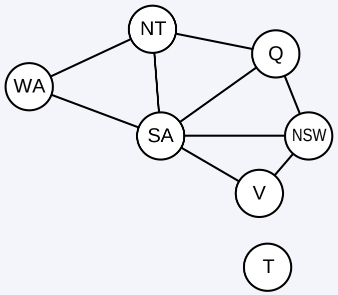
View of the Australian National University and Telstra Tower

One Solution



- Solution assignment:
  - $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$
  - Perhaps in addition $ACT = blue$

Constraint Graph



- a constraint graph can be used to visualize binary constraints
- for higher order constraints, hyper-graph representations might be used
- Nodes = variables, arcs = constraints

Note: Our problem is 3-colorability for a planar graph

Variations

- Binary, ternary, or even higher arity (e.g. ALL_DIFFERENT)
- Finite domains (d values) $\rightarrow d^n$ possible variable assignments
- Infinite domains (reals, integers)
  - *linear constraints*: solvable (in P if real)
  - *nonlinear constraints*: unsolvable

Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, ...)
- Spreadsheets
- Scheduling
- Floor planning
- Frequency assignments
- Sudoku
- ...

Backtracking Search over Assignments

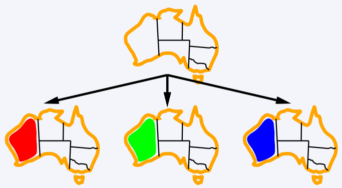
- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search
- Can solve n-queens for $n \approx 25$

Algorithm

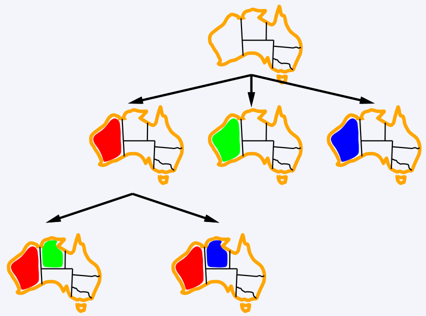
```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
     if value is consistent with assignment then
        add {var = value} to assignment
        inferences ← INFERENCE(csp, var, value)
        if inferences ≠ failure then
          add inferences to assignment
          result ← BACKTRACK(assignment, csp)
          if result ≠ failure then
            return result
     remove {var = value} and inferences from assignment
  return failure
```


Example (1)

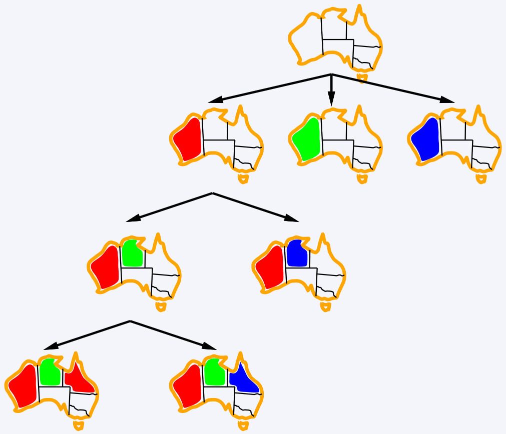
Example (2)



Example (3)



Example (4)



Improving Efficiency: CSP Heuristics & Pruning Techniques

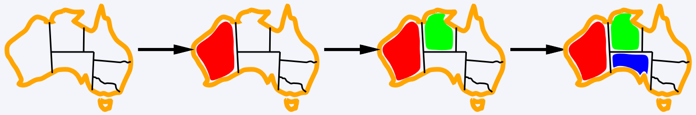
- Variable ordering: Which one to assign first?
- Value ordering: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure

→ Note: all this is not problem-specific!

Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values

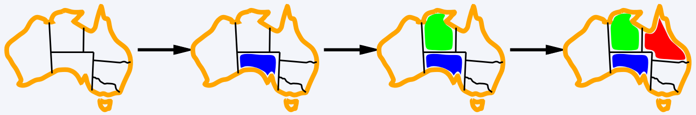
→ reduces branching factor!



Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables

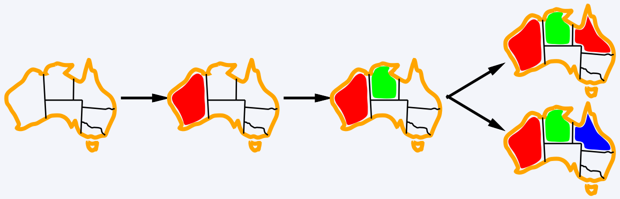
→ reduces branching factor in the next steps



Value Ordering: Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables

→ We want to find an assignment that satisfies the constraints (of course, does not help if unsat.)

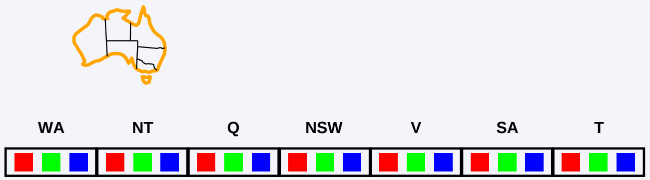


Rule out Failures early on: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed
- Implements what the ordering heuristics implicitly compute
- $WA = red$, then NT cannot become red
- If all values are removed for one variable, we can stop!

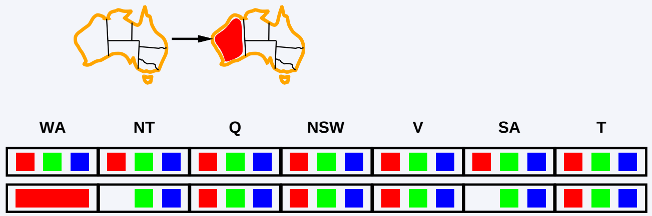
Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed



Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed



Forward Checking (3)

- Keep track of remaining values
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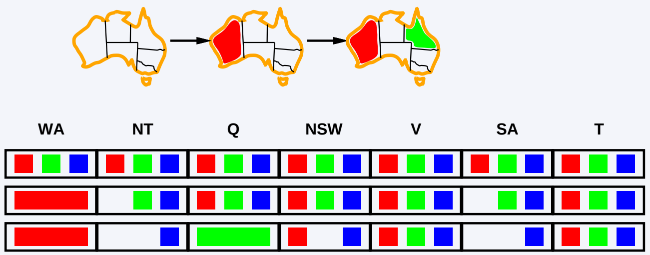
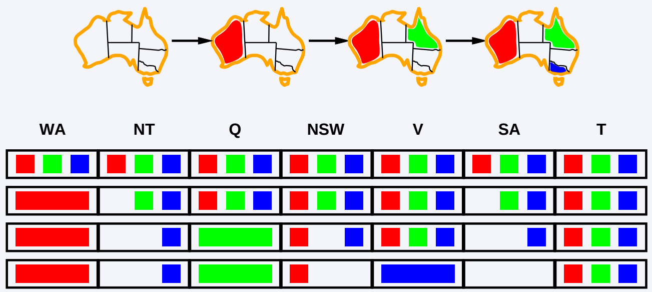


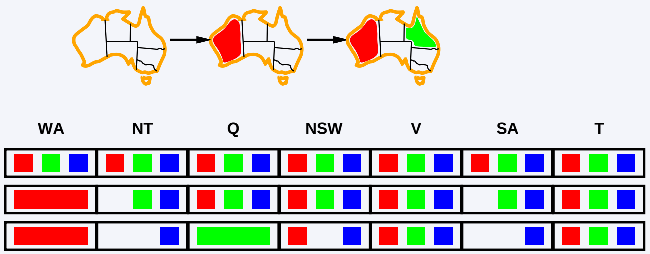
Forward Checking (4)

- Keep track of remaining values
- Stop if all have been removed



Forward Checking: Sometimes it Misses Something

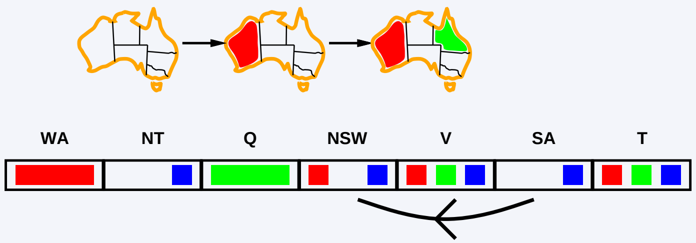
- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables



Arc Consistency

- A directed arc $X \to Y$ is "consistent" iff
  - for every value x of X, there exists a value y of Y, such that (x, y) satisfies the constraint between X and Y
- Remove values from the domain of X to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking

Arc Consistency Example



AC3 Algorithm

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components (X, D, C)
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REVISE(*csp*, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add (X_k, X_i) to *queue*
  return *true*

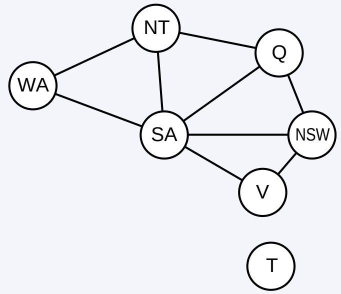
---

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised $\leftarrow$ *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised $\leftarrow$ *true*
  return *revised*

Properties of AC3

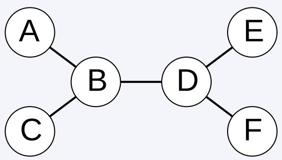
- AC3 runs in $O(d^3 n^2)$ time, with n being the number of nodes and d being the maximal number of elements in a domain
- Of course, AC3 does not detect all inconsistencies (which is an NP-hard problem)

Problem Structure (1)



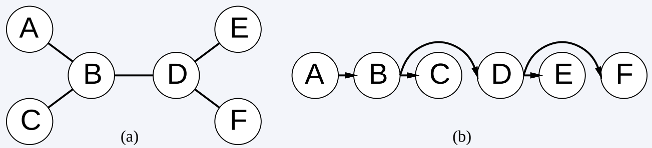
- CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically

Problem Structure (2): Tree-structured CSPs



- If the CSP graph is a tree, then it can be solved in $O(nd^2)$
  - General CSPs need in the worst case $O(d^n)$
- *Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root

Problem Structure (2): Tree-structured CSPs

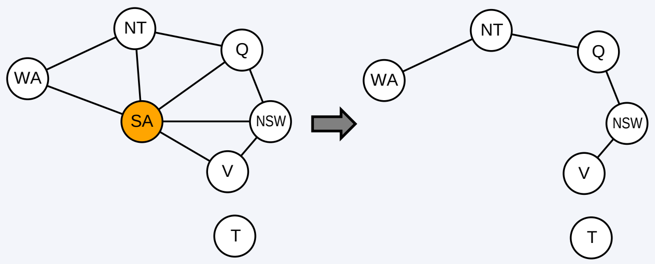


(a) (b)

- Pick any variable as root; choose an ordering such that each variable appears after its parent in the tree.
- Apply arc-consistency to (X_i, X_k), when X_i is the parent of X_k, for all $k = n$ down to 2.
- Now one can start at X_1 assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- Algorithm linear in n

Problem Structure (3): Almost Tree-structured

Idea: Reduce the graph structure to a tree by fixing values in a reasonably chosen subset

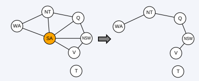


Instantiate a variable and prune values in neighboring variables is called Conditioning

Problem Structure (4): Almost Tree-structured

Algorithm Cutset Conditioning:

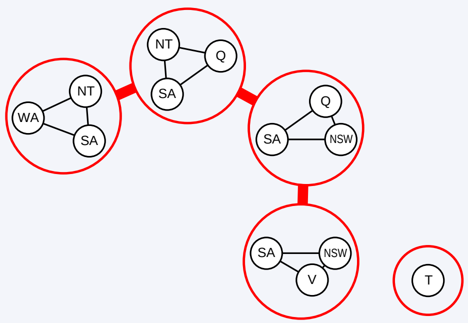
1. Choose a subset S of the CSPs variables such that the constraint graph becomes a tree after removal of S. S is called a cycle cutset.
2. For each possible assignment of varialbes in S that satisfies all constraints on S
   1. remove from the domains of the remaining variables any values that are inconsistent with the assignments for S, and
   2. if the remaining CSP has a solution, return it together with the assignment for S



Note: Finding the smallest cycle cutset is NP hard, but several efficient approximation algorithms are known.

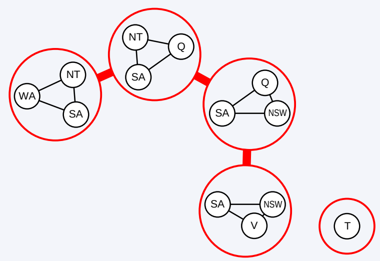
Another Method: Tree Decomposition (1)

- Decompose problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve sub-problems independently and combine solutions



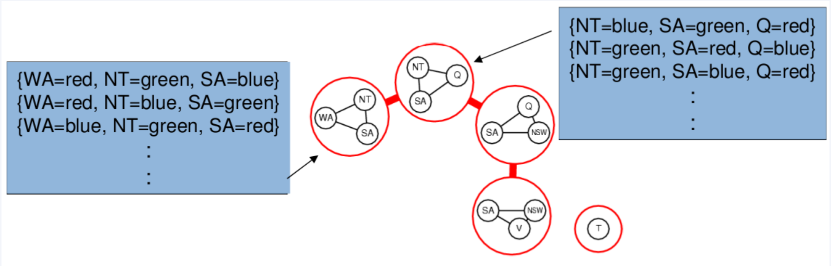
Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree



Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-variables, which have values defined by the solutions to the sub-problems
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable)



Tree Width

- The aim is to make the subproblems as small as possible. Tree width w of a tree decomposition is the size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width w and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$
- Unfortunately, finding a tree decomposition with minimal tree width is NP-hard. However, there are heuristic methods that work well in practice.

Summary & Outlook

- CSPs are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- Variable/value ordering heuristics can help dramatically
- Constraint propagation prunes the search space
- Path-consistency is a constraint propagation technique for triples of variables
- Tree structure of CSP graph simplifies problem significantly
- Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree
- CSPs can also be solved using local search